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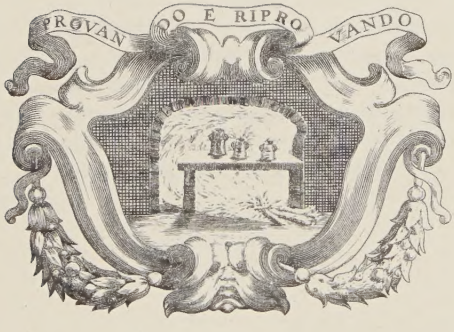

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On the Energy-Momentum Tensor of the Gravitional Field.

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(ricevuto il 5 Agosto 1960)

Summary. — In a previous note it was suggested that $\varkappa^{-1}\Lambda g_{\mu\nu}$, with $\Lambda > 0$ fulfills the basic requirements of a generally covariant energy-momentum tensor for the gravitational field. Some further justification is given for this hypothesis by *a*) studying the character of the field equations in the absence of the cosmological term; *b*) a re-interpretation of the line element based on Mach's principle; *c*) calculating the gravitational proper energy for de Sitter static universe with and without a Schwarzschild field present and showing that the difference is proportional to the Schwarzschild mass. Finally an attempt is made to define a true tensor for the angular momentum density. It is shown that in this way one is led to a definition of « physical co-ordinates ». These co-ordinates are determined in such a way that in the weak-field limit they approach the usual Lorentz co-ordinates. However it is not possible to arrive at these co-ordinates by imposing co-ordinate conditions, since they form the components of a vector. To the extent that the angular momentum of the gravitational field is covariantly conserved, the co-ordinates may be obtained as the gradient of a world scalar which satisfies an inhomogeneous d'Alembertian equation.

1. – Introduction.

In a previous note [1] several requirements were listed for the energy-momentum tensor of the gravitational field. It was pointed out that the cosmological term $\varkappa^{-1}\Lambda g_{\mu\nu}$ with $\Lambda > 0$, satisfied these requirements, and some of

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[1] F. R. TANGHERLINI: *Nuovo Cimento*, **15**, 385 (1960).

the main results based on using this tensor were briefly indicated. In this paper we shall discuss some of these results in greater detail and carry out the indicated calculations. We shall also attempt to enlarge on these results by considering the problem of defining a true angular-momentum density tensor in the general theory, this will lead us to define and discuss « physical co-ordinates » which are a covariant generalization of Lorentz coordinates.

Before carrying out these calculations, however, we should like to examine the problem of an energy-momentum tensor for the gravitational field in greater detail. Frequently the objection against a true tensor for the gravitational field is based on the principle of equivalence, namely: in a freely falling frame (*i.e.* one in which the $\Gamma^{\lambda}_{\mu\nu}$ vanish locally) the effects of the gravitational field disappear and hence presumably the gravitational energy momentum tensor. This forms the basis for the pseudo-tensors (or more accurately « complexes ») of EINSTEIN and LANDAU, although not that of MOLLER ([2]). However, strictly speaking, even in a freely-falling frame one does not eliminate the gravitational field but its gradients (*i.e.* the Newtonian « forces »), the special relativity metric still remains and bodies still have inertia. Now this persistency of the metric from the standpoint of Mach's principle is simply a consequence of the fact that by a co-ordinate transformation we can never eliminate a body's interaction with the distant masses of the universe, and hence the gravitational field ([3]). The expression we have given for the energy-momentum tensor, $\varkappa^{-1}\Lambda g_{\mu\nu}$ has the above features. In a locally Galilean frame the derivatives of the $g_{\mu\nu}$ vanish, even though $g_{\mu\nu}$ itself does not vanish but may be normalized to $\eta_{\mu\nu}$ (= diag. 1, −1, −1, −1) ([4]).

Let us now consider the invariance property of the mixed tensor $\varkappa^{-1}\Lambda g^{\mu}_{\nu}$. From a physical standpoint it may seem strange that an energy-momentum tensor should be an invariant. However as was also pointed out for this tensor, the pressure p satisfied identically the relationship $p = -u$, where u is the energy density.

Now let us consider the transformation properties of such an energy momentum tensor in special relativity (where by using the Minkowski formalism we can always eliminate the distinction between contravariant, covariant and

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([2]) For a recent review of conservation laws see J. G. FLETCHER: *Rev. Mod. Phys.*, **32**, 65 (1960), where further references are given. See also M. MAGNUSSON: *Mat. Fys Medd.*, **32**, 6 (1960).

([3]) Assuming that this interaction is in some way mediated by the gravitational field.

([4]) We shall employ a time-like metric here since g_{00}, g^{00} are positive definite as well as g^{0}_{0}. The use of such a metric then appears as a natural *consequence* of the requirement of positive-definiteness of the energy density of the gravitational field. Our use of a space-like metric in ([1]) was in order to facilitate comparison with the papers being discussed. However, in general, it is not possible to guarantee $g_{00} > 0$ everywhere (see Sect. 3), and it is not clear at present whether this implies such solutions should be modified, or whether these regions describe new, physically meaningful phenomena.

mixed tensors). We have in general for the energy momentum tensor

$$T^{\mu\nu} = (p + u)\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}s} - p\eta^{\mu\nu}. \tag{1}$$

Now setting $p = -u$, we have

$$T^{\mu\nu} = u\eta^{\mu\nu} \tag{2}$$

which is a special relativistic *invariant*. Finally if we adopt the Minkowski formalism ([5]) $\eta^{\mu\nu} \to \delta^{\mu\nu}$ and we have

$$T^{\mu\nu} = u\,\delta^{\mu\nu} (= \delta^{\mu}_{\nu} = \delta_{\mu\nu}) \tag{3}$$

for all observers, as indicated in the generally covariant formalism by $T^{\mu}_{\nu} = \varkappa^{-1}\Lambda g^{\mu}_{\nu}$. Thus we see that the invariance of δ^{μ}_{ν} in general relativity which follows on mathematical grounds has a physical analogue, even in special relativity, and represents a peculiar property of any substance for which $p = -u$.

Because of the invariance property of $\varkappa^{-1}\Lambda g^{\mu}_{\nu}$ we have argued previously that the homogeneous solutions to the field equations even if they have a wave-like nature do not transport gravitational energy. It also followed that in the limit $\Lambda \to 0$ there is no energy density in the gravitational field whatsoever. Because of the importance of this result, before proceeding further, we would like to establish it on more general grounds without employing the expression $\varkappa^{-1}\Lambda g_{\mu\nu}$.

2. – The vanishing of the gravitational energy in the absence of the cosmological term.

At first glance it might seem paradoxical that one should speak of a vanishing gravitational energy because of the well-known utility of the concept in Newtonian mechanics. For example a body of mass m raised a small distance d above the earth has in this formulation a gravitational potential energy mgd which may be then converted into work, heat etc... Surely in such circumstances we are witnessing the conversion of gravitational energy into other forms of energy! The answer we propose to this assertion on the basis of the field equations is « no »!. We arrive at this by the following observations: suppose the particle that was dropped has only weak interactions with matter, then the chances are that upon arriving at the surface of earth

([5]) Setting $x^{j} \to ix^{j}$ $(j = 1, 2, 3)$. We prefer this version to the more customary one of setting $x^{0} \to ix^{0}$, since the former keeps the energy real, which for some calculations is more convenient.

it would continue to pass right through, and there would be no transfer of energy. In other words the only experimental reason we have for believing in gravitational energy stems from the presence and collision of other fields, unlike for example electromagnetic energy for which we have direct experience in the form of energy-transmitting radiation (*).

By a process of idealization let us now consider particles that from the standpoint of special relativistic field theory have no interaction with each other at all. Clearly for such particles there will never be any collision that would result in a transfer of so-called gravitational energy into other forms of energy. Falling together via the gravitational field, they will pass through one another. But they will be completely unaware of each other's existence or motion. This is because there is no way such information could be imparted, since, by assumption associated with each particle is only one physical number, its mass and as we shall now show it is always possible to choose the co-ordinate system for point particles so that this number is a constant of the motion for an observer travelling with the particle.

Moreover this result will be deduced without introducing a pseudo-tensor for the gravitational field, since as we shall now see with appropriately chosen co-ordinate conditions (6) the covariant conservation law

$$\frac{\partial \sqrt{-g}\, T_{\nu}^{\mu}}{\partial x^{\mu}} - \frac{1}{2} \frac{\partial g_{\lambda\varrho}}{\partial x^{\nu}} T^{\lambda\varrho} \sqrt{-g} = 0 \,, \tag{4}$$

is sufficient. That is to say, for particles, under the assumptions below, it represents a true conservation law.

(*) *Note added in proof.* – Needless to say, we are not denying the utility of the concept of gravitational potential energy, but rather emphasizing that this quantity should not be regarded as due to a distribution of gravitational energy *per se* as is the case for the electromagnetic interaction. For example, for the Schwarzschild field, the Einstein pseudo-tensor vanishes in quasi-rectangular co-ordinates, whereas for the electromagnetic field, for the comparable case of a charged particle, the Maxwell energy-momentum tensor does not vanish, and indeed is a true tensor. To the extent we wish to use the concept of gravitational potential energy, we have in a static co-ordinate system for a particle of mass m instantaneously at rest in the field, $U = m\frac{1}{2}(g_{00} - 1)$. It is from this *interaction energy* that the concept has arisen of gravitational energy existing as an independently distributed quantity, *i.e.*, « another field » analogous to the electromagnetic field. Our position is that only for non-vanishing cosmological term can the concept of gravitational energy as a physical quantity be consistently justified, and even then, as indicated in Sect. 3, with some restrictions on $\varkappa^{-1}\Lambda$, if the final result is to be compatible with Mach's principle.

(6) This reliance on co-ordinate conditions we hope to eliminate or make more plausible in a future development, perhaps along the lines indicated in Section 4. Alternatively this method may be looked upon as (partially) choosing the co-ordinates for the integration problem. If the restrictions are not too severe so as to destroy the generality of the solution, we can then find the solution (*i.e.* the metric) in any other co-ordinate system by performing a co-ordinate transformation.

First of all we note that since we are dealing with particles, if we assume that they move without collision, we can in the neighborhood of a given particle disregard the contribution to $\sqrt{-g}\,T^{\mu}_{\nu}$ from the other particles. Provided, of course, the particle is conserved as a particle which we shall show is the case. Let us now focus attention on a given particle and choose our co-ordinate system such that the particle is at rest $\mathrm{d}x^i/\mathrm{d}s = 0$, *throughout the dynamical development*, our final co-ordinate condition will guarantee this. Now under these circumstances the above conservation law takes the simple from (*)

$$\frac{\partial}{\partial x^0}\sqrt{-g}\,\varrho(x^{\nu})\,\frac{g_{0\mu}}{g_{00}} = \frac{1}{2}\,\frac{\partial g_{00}}{\partial x^{\mu}}\,\frac{\sqrt{-g}}{g_{00}}\,\varrho(x^{\nu})\,, \tag{5}$$

where $\varrho(x^{\nu})$ is the scalar density, which in a proper co-ordinate system at $x^0 = 0$, say, was a δ-function. The problem is whether this δ-function is conserved in time. Now let us choose as our fourth co-ordinate condition $\partial g_{00}/\partial x^0 = 0$. Then (5) becomes

$$\begin{cases} \dfrac{\partial\sqrt{-g}\,\varrho}{\partial x^0} = 0\,, \\ \sqrt{-g}\,\varrho\,\dfrac{\partial g_{0i}}{\partial x^0} = \dfrac{1}{2}\,\dfrac{\partial g_{00}}{\partial x^i}\,\sqrt{-g}\,\varrho\,. \end{cases} \tag{6}$$

From the first equation it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\sqrt{-g}\,\varrho\,\mathrm{d}^3x = 0\,. \tag{7}$$

So that indeed the energy $= \int\sqrt{-g}\,\varrho\,\mathrm{d}^3x$ is a constant of the « motion », moreover if $\sqrt{-g}\,\varrho$ was a spatial δ-function at $x^0 = 0$, it remains so throughout the motion. The other three equations are a requirement on the co-ordinate system if the conservation law is to be satisfied (they also have another interpretation which we shall discuss shortly). We may therefore summarize the co-ordinate conditions as

$$\frac{\partial g_{0\mu}}{\partial x^0} = \frac{1}{2}\,\frac{\partial g_{00}}{\partial x^{\mu}}\,. \tag{8}$$

(*) *Note added in proof.* – Assuming that for a particle at rest, $T^{00} = \varrho/g_{00}$ with the other terms vanishing, as suggested by the tensor for « dust »: $T^{\mu\nu} = \varrho(\mathrm{d}x^{\mu}/\mathrm{d}s)(\mathrm{d}x^{\nu}/\mathrm{d}s)$. However, in the case of the Schwarzschild field, this only holds in the so-called weak field approximation with $\mathrm{d}s^2 = (1-(2Gm/r))\,\mathrm{d}t^2 - (1+(2Gm/r))\,\mathrm{d}x^i\,\mathrm{d}x^i$. Thus one has $\Box(h^{\nu}_{\mu} - \frac{1}{2}\delta^{\nu}_{\mu}h) = -2\varkappa T^{\nu}_{\mu}$, with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and $h^{\nu}_{\mu} = \eta^{\nu\lambda}h_{\lambda\mu}$, $h = h^{\mu}_{\mu}$. Substituting the above values of $h_{\mu\nu}$ one finds readily only $T^0_0 \neq 0$. But we found recently (*Phys. Rev. Lett.* **6**, 147 (1961)) that the exact structure of T^{ν}_{μ} is different: only the trace of the spatial stresses vanishes. See note at end of this section.

Let us now substitute these conditions in the geodesic equations to see whether they are consistent with our assumption $dx^i/ds = 0$, we find, bearing in mind only terms such as $\Gamma^{\nu}_{00}(dx^0/ds)^2$ contribute,

$$\frac{d^2x^\nu}{ds^2} + g^{\nu\mu}\left[\frac{\partial g_{\mu 0}}{\partial x^0} - \frac{1}{2}\frac{\partial g_{00}}{\partial x^\mu}\right]\left(\frac{dx^0}{ds}\right)^2 = 0\,, \tag{9}$$

or $d^2x^\nu/ds^2 = 0$, so that the particle remains unaccelerated and hence the conditions are consistent. Thus for particles there is nothing in the field equations or the dynamics which corresponds to a transfer of energy from one particle to another. There is therefore no way of informing one particle of the motion of another, unless one introduces interactions in the sense of special relativity which correspond to distributed fields and since such fields are not incorporated in the T^μ_ν used above, their introduction is incompatible with the problem in question.

Typical examples of the above co-ordinate conditions are provided by the choice of $g_{0\mu} = \delta_{0\mu}$ as employed in cosmology, or in Lemaitre's non-static expression for the Schwarzschild field. However the above conditions are more general and are satisfied by any choice of co-ordinates for which

$$\begin{cases} g_{00} = 1 + 2\,U(x^i)\,, \\ g_0 = \dfrac{\partial U}{\partial x_j}\,x^0 + V_j(x^i)\,, \end{cases} \qquad (i, j = 1, 2, 3), \tag{10}$$

where we have written the expression for g_{00} in the above form for the following reason: if we choose our spatial co-ordinate system appropriately the conditions (8) may be looked upon as a disguised form of Newton's equations of motion in a gravitational field. However, since the particle is taken to be at rest $dx^i/ds = 0$, it is the space-time structure rather than the particle that undergoes the acceleration, regarding g_{0i} as a generalized velocity. Hence the above co-ordinate conditions represent a kind of Ptolemaic version of the Newtonian equations, it is in keeping with the spirit of general relativity and Fermi's theorem that such a version should be possible.

We have gone through the above discussion in perhaps unnecessary detail to bring out the fact that the concept of gravitational pseudo-energy serves no useful purpose whatsoever in the discussion of the motion of mass points and also that there is no mechanism in the general theory for a system mass points to emit and absorb energy. This is in contrast with a system of charges in electromagnetic theory. The analogy between the two disciplines being that just as electromagnetic waves do not transport charge, gravitational waves do not transport energy. The Bianchi identities conserves energy of mass points just as the asymmetry of $F_{\mu\nu}$ conserves charge, $\partial^2\sqrt{-g}F_{\mu\nu}/\partial x^\mu \partial x^\nu \equiv 0$.

Needless to say it would be desirable to supplement the above discussion with an *exact* solution to the two body problem, say in the case of two mass points which are oscillating back and forth along their line of centers by passing through one another. It would be natural for this problem to choose the initial conditions so that both particles were at rest. Then, adopting the above co-ordinate conditions they would remain at « rest » (7) throughout the dynamical development, the motion of course being transferred to the non-static form of the line-element. Since $\mathrm{d}/\mathrm{d}t\int\sqrt{-g}\,T_0^0\,\mathrm{d}^3x = 0$, there can be no question of gravitation radiation removing energy from the system. If further we choose $g_{00} = 1$ (8), then also $\mathrm{d}/\mathrm{d}t\int\sqrt{-g}\,T_i^0\,\mathrm{d}^3x = 0$ corresponding to the conservation of momentum. Under these circumstances $g_{0i} = f_i(x^j)$.

On the basis of the above discussion it should be possible to find an exact solution to the problem which exhibits stationary states.

The assumption of perfect transparency for the particles seems the only justifiable one in view of the inherent relativistic difficulties for defining, say « hard spheres ». In any case the rigorous solution of the field equations may suggest in a natural way the correct condition on the scattering. Thus it may be, that just as the field equations contain the Newtonian and post-Newtonian equations of motion of point particles under the assumptions of the Einstein, Infeld, Hoffmann method (9), so with appropriate assumptions they may contain the scattering as well (*).

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(7) This follows also from Newton's first law when we take into account that the gravitational field does not exert true « forces » from the standpoint of the general theory.

(8) Strictly speaking all we need is $g_{00} = 1$, $\partial g_{00}/\partial x^i = 0$ at each particle, and hence $\partial g_{0i}/\partial x^0 = 0$ at each particle. These are our only « boundary conditions ». As Einstein pointed out in the early days, boundary conditions at infinity are entirely gratuitous and are extremely complicated assumptions.

(9) For a review of this work, see, *e.g.* L. Infeld: *Rev. Mod. Phys.*, **29**, 398 (1957).

(*) *Note added in proof.* – Because of the more complicated structure we have found recently for the energy-momentum tensor of a point particle, the above argument must be revised. Thus for a point mass source for the Schwarzschild field, we have found, in polar coordinates, $T_\nu^\mu = \mathrm{diag}\,(1, 1, -\frac{1}{2}, -\frac{1}{2})\,T$, where $T = T_\mu^\mu = m\delta(r)/2\pi r^2$. Hence, for a particle at rest, although the spatial trace T_i^i vanishes, the individual spatial components in general do not. We therefore have a weaker relation than for « dust », *i.e.*, for a point particle at « rest »: $T_0^0 = T$, $T_i^i = 0$, and this is of course an invariant statement under subsequent purely spatial transformations, *e.g.*, in quasi-rectangular co-ordinates, $T_0^0 = T$, $T_j^i = \frac{1}{2}T(3n_in_j - \delta_{ij})$, where $n_i = x^i/r$.

However, under these circumstances, we cannot legitimately use a representation for a particle given by $T^{\mu\nu} = T(\mathrm{d}x^\mu/\mathrm{d}s)(\mathrm{d}x^\nu/\mathrm{d}s)$, since by definition a particle is « at rest » if $\mathrm{d}x^i/\mathrm{d}s = 0$, and this would imply $T^{ij} = 0$, which is not the case. Thus some new representation is needed for the particle's tensor which will be an appropriate generalization of the classical one. The following remarks are a very preliminary and tentative approach to this question.

For a particle, we define a « structure tensor » with the following properties: *a*) $T^{\mu\nu} = T\Phi^{\mu\nu}$ (T is assumed to be a singular distribution); *b*) $\Phi^{\mu\nu} = \Phi^{\nu\mu}$; *c*) $\Phi_\mu^\mu = 1$;

3. – Calculations with the cosmological term and Mach's principle.

The historical reason for introducing $\Lambda g_{\mu\nu}$ was that EINSTEIN wanted to construct a closed static universe which would at once reflect Mach's principle and the (at that time) static properties of the fixed stars. However as was shown by de Sitter there exists a solution even in the absence of matter:

$$ds^2 = \left(1 - \frac{r^2}{R^2}\right) dt^2 - g_{00}^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) , \tag{11}$$

so that the introduction of $\Lambda g_{\mu\nu}$ does not guarantee Mach's principle. Moreover as the red shift of the distant galaxies indicate, the universe is not static. As a consequence of these results together with Friedmann's solutions, EINSTEIN later argued that there was no longer any justification for the term ([10]).

The difficulty that remains, however, is how are we to incorporate Mach's principle into the general theory? On the other hand the following approach

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d) for a particle « at rest »: $\Phi_0^i = \Phi_i^0 = 0$ $\Phi_i^i = 0$ (and hence $\Phi_0^0 = 1$). Thus we have seven conditions on the particle's structure tensor for « rest » instead of the classical nine, which would be $\Phi^{i\mu} = 0$. The imposition of two less constraints on the tensor for rest are required by the fact that only the trace of the spatial stresses vanishes.

Let us now adopt co-ordinates in which the particle is at rest; this will in general require three co-ordinate conditions. From $T^{\mu}_{0;\mu} = 0$, we shall then obtain the conservation statement $\partial_0 \sqrt{-g}\, T^0_0 = 0$, provided we can satisfy at the position of the particle $\Phi^{\mu\nu}\partial_0 g_{\mu\nu} = 0$, or, alternatively, since $\Phi^{\mu\nu} g_{\mu\nu} = 1$, we have $g_{\mu\nu}\partial_0 \Phi^{\mu\nu} = 0$. In this form, the requirement appears as a condition on the time development of the particle's structure tensor when it is at rest.

In the case of the gravitational energy momentum tensor, $E^{\mu}_{\nu} = \varkappa^{-1}\Lambda \delta^{\mu}_{\nu}$ since E^i_0, E^0_i vanish identically, the gravitational field is, so to speak, always at rest for all observers in all coordinate systems — as we would expect from the basic principles of relativity. However, to obtain a conservation statement, one must impose the condition $\partial_0 \sqrt{-g} = 0$, or $g_{\mu\nu}\partial_0 g^{\mu\nu} = 0$, in analogy with the particle. However, in this case it is a genuine coordinate restriction, and restricts the definition of time by a conservation requirement. Since in general we require $\sqrt{-g} > 0$ this restriction will then be guaranteed throughout the dynamical development.

In order to guarantee that the conservation condition $\partial_0 \sqrt{-g} = 0$ will be compatible with the particle conservation condition $g_{\mu\nu}\partial_0 \Phi^{\mu\nu} = 0$, we define the vector $S_\nu = \Phi^{\mu}_{\nu;\mu}$ and let S_ν be space-like $S_\nu S^\nu < 0$. Then for a particle at rest we demand $S_0 = 0$, and hence $\partial_0 \sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\partial_0 \Phi^{\mu\nu}$ from which the compatibility follows.

However, the above remarks should be looked upon as only a temporary expedient in place of a rigorous argument from first principles leading to a well-defined representation for the T^{μ}_{ν}. The structure of the cosmological term has proven to be useful in imposing suitable requirements on the T^{μ}_{ν} by analogy, but as we shall see below, the introduction of the cosmological term is not free of difficulties when we inquire as to the total energy of the gravitational field.

([10]) A. EINSTEIN: *Sitzber. Preuss. Akad. Wiss.*, **12**, 235 (1931); see, however, H. P. ROBERTSON: *Helv. Phys. Acta Supp.*, **4**, 128 (1956). see also W. A. BAUM: *Astron. Journ.* **62**, 6 (1957); A. SANDAGE; *Astrophys. Journ.* **127**, 513 (1958).

to Mach's principle leads immediately to the cosmological term. If we demand that inertial effects arise as a consequence of a body's motion relative to the distant masses, then effectively we are saying there would be no particle dynamics if there were no distant masses. Now the simplest way to insure this would be to construct a particle dynamics on the principle

$$\delta \int \sqrt{E_{\mu\nu}\, \mathrm{d}x^{\mu}\, \mathrm{d}x^{\nu}} = 0 \,, \tag{12}$$

where $E_{\mu\nu}$ is a mean energy-momentum tensor for the background field, if $E_{\mu\nu} = 0$, clearly there is no dynamics. Now we construct a geometry on $E_{\mu\nu}$, and then use Einstein's field equations to write

$$G_{\mu\nu}(E_{\mu\nu}) = -\varkappa E_{\mu\nu} \,, \tag{13}$$

which we recognize as the field equations with cosmological term in units for which $\varkappa^{-1}\Lambda = 1$ and the singularities have been smeared out into the $E_{\mu\nu}$. On the other hand, we know observationally that matter is not smeared out and so there is still some mystery as to how a particle is aware of the existence of the other particles. One answer lies in assuming that there is a continuous distribution of energy associated with the gravitational field and that it is this quantity which determines the metric, the material particles and other forms of energy acting as a perturbation on this distribution, *i.e.*

$$G_{\mu\nu}(E_{\mu\nu}) = -\varkappa(E_{\mu\nu} + T_{\mu\nu}) \,. \tag{14}$$

Thus $g_{\mu\nu}$, or in ordinary units $\varkappa^{-1}\Lambda g_{\mu\nu}$, should be looked upon as the energy-momentum tensor for the gravitational field.

Alternatively we may regard the assumption $\varkappa = \Lambda$ as not merely a convenient choice of units but a natural one if we consider the possibility that the cosmological constant and the gravitational constant (*i.e.* $8\pi G/c^4$) are not really two different constants of nature, but the same, and their apparent difference is due to an initial unfortunate choice of units. Clearly it is only reasonable to construct a system of units upon invariant quantities of the problem, *i.e.*, E_0^0, and c. In this system of units we have the following table:

Quantity	Dimensions
Fermi interaction coupling constant	L^6
Action, angular momentum	L^4
Energy, mass	L^3
Force, charge	L^2
Electromagnetic potentials, meson field	L
Velocity, energy density, pressure, electromagnetic field strengths	L^0
Acceleration, charge density, current	L^{-1}
Gravitational constant, gauge scalar, Δ-functions	L^{-2}

Although integral powers of a length [11] are obtained also for units with $\hbar = 1$, $c = 1$, or $G = 1$, $c = 1$, etc., the advantage of the above system is that it associates, for example, mass with a volume, in keeping with our elementary definition of mass. The association of force with a cross-section is not surprising if we consider that in a scattering process we change the momentum of the particle by ΔP in the interaction time τ, that is in classical language we exert a force on the particle and this is a measure of the cross-section. Of more interest for our purpose here, however, is the association of the vector potential with the dimensions of co-ordinates which we shall employ as a guide later on.

We shall now use the expression for the energy-momentum tensor to calculate the proper energy of the gravitational field in the case of no particles present, de Sitter static universe, and in the case of one particle present (*).

For the former case we have an expression of the form in polar co-ordinates

$$E_0 = \varkappa^{-1} \Lambda \int \sqrt{-g}\, g_0^0 \mathrm{d}^3x = 4\pi\varkappa^{-1}\Lambda \int r^2 \mathrm{d}r \,. \tag{15}$$

However, a curious difficulty arises in connection with assigning the domain for this integral. If we argue that, since for $r > R$, $g_{00} < 0$, $r = R$ represents a natural cut-off, then we should be similarly obliged to perform such a cut-off at the lower limit that occurs when the Schwarzschild field is present, *i.e.*, neglecting the cosmological term, at $r = 2Gm$. But it would then follow, for a point mass, since $T_0^0 = m\,\delta(r)/2\pi r^2$ that

$$E\,(\text{Schwarzschild}) = 4\pi \int_{2Gm} T_0^0 r^2 \mathrm{d}r = 0 \,. \tag{16}$$

However, mathematically, it is quite difficult to justify such a cut-off, since the Riemann invariants for the Schwarzschild field, given by $R^{\lambda\mu}{}_{\nu\sigma}$, behave as $1/r^3$ and are indeed regular at $r = 2Gm$, showing only a singularity at $r = 0$. Also the field equations $G_\nu^\mu = -\varkappa T_\nu^\mu$ show no barrier at $r = 2Gm$.

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(11) Except of course for the Schrödinger wave function which has dimensions $L^{-\frac{3}{2}}$ (as well as the spinor field). In this dimensional approach we see in a simple way the non-classical character of ψ, since all other « physical » quantities are associated with integral dimensions. One might consider a redefinition of ψ by regarding it as the amplitude for finding the inertia of a particle, in which case ψ would be dimensionless. Apart from dimensional considerations, the basis for such an approach would lie in the fact that the problem of inertia is not dealt with in the foundations of quantum mechanics. This was natural in the early days of the theory but at present it can only be regarded as a serious deficiency which becomes all the more acute when we attempt to understand the mass spectra of elementary particles from first principles.

(*) Presentation in what follows altered in proof.

Indeed, we should be confronted with a contradiction in attempting to integrate the field equations for a point mass to obtain the Schwarzschild-solution, since

$$g^{11} = -1 + \frac{8\pi G}{r}\int_0^r T_0^0 r'^2\,\mathrm{d}r' .$$

(Actually there would be no difficulty if $m<0$, *i.e.*, both negative rest mass and gravitational mass, but such particles have never been observed.) Thus in order to avoid a contradiction in general relativity, one must assume either:

A) There are no point masses ($m>0$), or more generally, there are no particles whose mass distribution is sufficiently singular so that they develop a Schwarzschild « shell »; hence, for exampler, one would require $r\to 0$, $T_0^0 \to Ar^{-2}+Br^{-1}lnr+Cr^{-1}$ $(A>0)$; or,

B) The quantities g_{00}, g_{41} can change sign or « flip ». (The signature is still of course -2; the more difficult question of whether or not the signature can change will not be discussed here.)

Now, customarily, one assumes *B*) to be false in general relativity, whence *A*). Unfortunately, we know of no rigorous proof that *B*) cannot occur. We believe it to be true from our experience in the macroscopic realm, but that it should hold at very small distances (or even very large) is an hypothesis. On the other hand, if we assume *B*) to be true, the general theory becomes remarkably simple for the case of a point mass, *e.g.*, T is a δ-function, and the gravitational action is

$$A = \int \varkappa^{-1}\sqrt{-g}R\,\mathrm{d}^4x = \int \sqrt{-g}T\,\mathrm{d}^4x = m(t-t_0) ,$$

where m is the Schwarzschild mass. Hence gravitational mass and the rest mass are equal, and the problem of gravitational « clothing effects » does not arise. However, if *B*) holds, it would also imply $E_0=\infty$. For the case with one particle present, corresponding to the line element

$$\mathrm{d}s^2 = \left(1-\frac{2Gm}{r}-\frac{r^2}{R^2}\right)\mathrm{d}t^2 - g_{00}^{-1}\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2) , \tag{17}$$

we would have

$$E_1 = 4\pi\int_0^\infty T_0^0 r^2\,\mathrm{d}r + 4\pi\int_0^\infty \varkappa^{-1}\Lambda r^2\,\mathrm{d}r , \tag{18}$$

and only the difference energy would be finite, $\Delta E = E_1 - E_0 = m$. This constitutes one possible argument against either B) or against the cosmological term, although to the extent we work with energy differences such a divergent « shift » does not appear to pose any serious difficulties.

If we assume B) false, then the Schwarzschild « shell » does not occur, *i.e.*, (17) does not hold at small distances, and some such energy momentum tensor as described in A) holds as $r \to 0$, so that $2Gm/r$ becomes replaced by a suitable « interior » solution. While, to the extent we keep the cosmological term we must cut-off the integral (15) at $r = R$ and one has [12] ($R^2 \equiv 3/\Lambda$),

$$E_0 = 4\pi\varkappa^{-1}\Lambda\int_0^R r^2\,\mathrm{d}r = G^{-1}R/2\,. \tag{19}$$

While for (17), the cut-off is provided by the largest root e_1 of the cubic $g_{00} = 0$, *i.e.*,

$$e_1 = R - Gm + 0(G^2m^2/R) \tag{20}$$

and we obtain for E_1,

$$E_1 = m + \frac{G^{-1}R}{2} - \frac{3}{2}\,m + 0(Gm^2/R)\,. \tag{21}$$

As previously indicated [1], we see that E_1, $E_0 \to \infty$, as $\Lambda \to 0$, but the difference is finite and given by $E_1 - E_0 = -\frac{1}{2}m$.

Thus the effect of a particle on the gravitational field would be to diminish the allowed « volume » of space and to *lower* the energy of the system. The system would therefore be unstable in the direction of producing particles, reminiscent of continuous-creation type theories. It is also interesting to see how the gravitational field energy would exhibit a typically *strong-coupling* type of series, *i.e.*, terms going as G^{-1}, ..., G, etc.

Nevertheless, it seems quite unsatisfactory to cut-off the integral in this way without a suitable cosmological structure to justify it. However, when one takes into account the other matter in the universe (as discussed below), it may lead to a finite universe (*e.g.* « spherical ») which would then, effectively, provide a cut-off.

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[12] One should distinguish between this « cut-off universe » and one of the Einstein type for which the comparable integral is of the form $4\pi\int r^2(1 - r^2/R^2)^{-\frac{1}{2}}\,\mathrm{d}r$ with either an elliptic or spherical-space interpretation. Indeed if we make the substitution $r = R\sin\chi$ in (19), with $0 \leq \chi \leq 2\pi$ for the spherical interpretation, the integral vanishes, but then $\sqrt{-g}$ changes sign.

In view of the above difficulty raised by the cosmological term, one might ask whether or not it is really necessary to introduce it in order to obtain an expression of Mach's principle in the general theory, and does not the theory already satisfy the principle? To be sure, one might also ask whether one should try to satisfy the principle?

With respect to the latter question, we observe that for a theory of relativity, something like Mach's principle is necessary in order that the relativity of velocity contained in the special theory, and the relativity of acceleration contained in the principle of equivalence appear as a natural consequence of the basic assumptions and not as something artificial. Only if inertia is a collective phenomenon, as implied by Mach's principle, is such the case.

Returning to the former question, we note that while it is unquestionably true that the general theory does exhibit many features strongly suggestive ([13]) of Mach's principle, there are still basic difficulties that have never been satisfactorily answered. For example, the Einstein, Infeld, Hoffmann method enables one, by approximation techniques, to obtain the equations of motion for two bodies in a universe otherwise free of matter, and in lowest order these equations are nothings but Newton's equations for gravitating bodies. The masses of the two bodies in this approximation may be assigned independently and arbitrarily. Moreover, by assumption, there are no other bodies in the system. This therefore implies that there is nothing in the field equations that would demand the existence of the fixed stars in order to arrive at ord nary dynamics. Thus if we imagine the two bodies to be circling about one another, the Newtonian equations of motion imply the existence of inertial forces balancing the gravitational force—even though there is no « distant matter » in the field equations to produce these forces.

A partial answer lies in the fact that one imposes flat space boundary conditions in solving the field equations: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $h_{\mu\nu} \to 0$ at infinity, and it is frequently argued that in some way this is accounting for the distant masses. This is clearly an unsatisfactorily argument, since in the absence of the cosmological term, $\eta_{\mu\nu}$ is a rigorous solution of the field equations with $T^{\mu}_{\nu} = 0$. If one claims $\eta_{\mu\nu}$ is the basic inertial « field », then it can exist in the absence of matter. One has therefore arrived at an inconsistency.

It is primarily for this reason we have therefore attempted to reintroduce the cosmological term (following Einstein's original arguments), treating now $\varkappa^{-1}\Lambda\delta^{\mu}_{\nu}$ as the energy-momentum tensor of the gravitational field E^{μ}_{ν}. (Another reason is that the constants G, c, do not yield a unit of mass.) The basic

([13]) W. Davidson: *Monthly Notices Royal Astron. Soc.*, **117**, 212 (1957). On the other hand see also D. W. Sciama: *Monthly Notices Royal Astron. Soc.*, **113**, 34 (1953). Also the original formulation of the problem: A. Einstein: *Sitzber. Preuss. Akad. Wiss.*, **6**, 142, (1917).

flat-space normalization would then arise in the following way. Since E^{μ}_{ν} is an invariant, this tensor does not fully define the space, therepre one has to imposesome conditions on $E_{\mu\nu}$, and the condition we impose is that in the neighborhood of a point removed from matter, $E_{\mu\nu}$ has a structure corresponding to the infinitesimal propagation of light signals (*i.e.* hyperbolic structure). Thus $E_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}$ is required to have real solutions for the displacements—and as indicated previously, we may use the $E_{\mu\nu}$ to define a geometry. This specification of $E_{\mu\nu}$ via an « interaction » with light, together with the field equations (which relate $E_{\mu\nu}$ to E^{μ}_{ν} for an arbitrary system of co-ordinates, $G^{\mu}_{\nu}(E_{\mu\nu}) = -\varkappa E^{\mu}_{\nu}$) leads to a metric that has the required property, upon suitably choosing the origin of the co-ordinate system to be the point in question (de Sitter metric). Thus if matter is localized to a « small » region, and if we do not go to very large regions, such an approach is analogous to choosing flat-space boundary conditions.

However such an approach to Mach's principle is still not satisfactory, since the distant masses could then not be directly held responsible for the inertial effects in, *e.g.* a rotating frame, but rather they would arise because the effects of a rotation cannot be eliminated from the $E_{\mu\nu}$ (thought of now as $\eta_{\mu\nu}$) by subsequent co-ordinate transformations which preserve the rotation, unlike translations. To be sure, since material bodies would perturb the $E_{\mu\nu}$, they would be expected to produce via this perturbation, $\Delta E_{\mu\nu}$, effects *entirely similar* to inertial effects, and this is indeed the case both theoretically and experimentally. But it is still not satisfactory. Only if $\varkappa^{-1}\Lambda$ itself could be related to the mean density of matter in the universe, say as

$$\varkappa^{-1}\Lambda \propto \int T\sqrt{-g}\,\mathrm{d}^3x \Big/ \int \sqrt{-g}\,\mathrm{d}^3x\,, \tag{22}$$

would the principle be satisfied, since then matter and gravitation field could not exist independently. A more thorough-going analysis would then have to show that actually the de Sitter universe would be unstable against the production of matter (as our preceding calculation would suggest) which could then appear and be reabsorbed until some equilibrium were reached and the above relation satisfied. Under these circumstances what we have called the energy-momentum tensor of the gravitational field would alternatively represent an energy-momentum tensor for some new state of matter, in many respects analogous to a superfluid.

At present, we know of no way by which such relations as (22) may be derived in a rigorous fashion taking into account the non-static properties of the universe, and in such a way as to bring us a little closer to understanding how these rather fundamental concepts are related to the necessity for a quantum theory, and how the elementary particles manage to acquire intrinsic

angular momentum and rest mass. From Mach's principle, one would expect $\hbar \propto \varkappa^{-1}\Lambda$, $m \propto \varkappa^{-1}\Lambda$, but at present we can only give dimensional arguments to arrive at such relations (1). It is perhaps satisfying, however, from the standpoint of Mach's principle, to see that a theory that requires an intrinsic relation of inertia to cosmology automatically supplies a length of $\sim 3 \cdot 10^{-13}$ cm $(l = (G\hbar R)^{\frac{1}{2}} c^{-1})$, whereas a theory based on G, $\hbar$, c leads to a much smaller length $\sim 10^{-33}$ cm. On the other hand, the question becomes further complicated by the obviously important role the electromagnetic field must eventually play in these considerations, *e.g.*, with respect to angular momentum conservation in elementary processes. We therefore examine below the question of formulating a true angular momentum tensor for the gravitational field.

4. – Angular momentum density of the gravitational field and the problem of defining physical co-ordinates which are covariant quantities.

Although the preceding development indicates that we can define an energy tensor for the gravitational field which has reasonable properties, we encounter a very fundamental difficulty in the general theory when we consider the problem of defining an angular momentum tensor $L^{\lambda\mu\nu}$.

This problem moreover arises independently of whether it is the angular momentum tensor for gravitation or for matter. It is a consequence of the fact that we need an expression of the form $T^{\lambda\nu}x^{\nu} - T^{\nu\nu}x^{\mu}$, and such an expression has no meaning in the general theory since the co-ordinates employed there do not transform as the components of a vector (and therefore have no physical significance). Now as a consequence we may look upon angular momentum density as something that has meaning only in special relativity, or we can attempt to develop a theory of co-ordinates which will give rise to a true tensor $L^{\lambda\mu\nu}$ and meet the full requirements of mathematical covariance. The following development, although preliminary, suggests the utility of such an approach.

Let us consider for simplicity, the problem of constructing a gravitational field angular momentum tensor. Then we desire an expression similar to $g^{\lambda\mu}x^{\nu} - g^{\lambda\nu}x^{\mu}$, with however the x^{μ} replaced by a true vector that locally reduces to x^{μ}. Now the only vector field we have available in our system is the electromagnetic field which in our units also has the dimensions of a length. Although this is not quite what we want we shall proceed as though it were and this will indicate a solution. We therefore form the tentative expression

$$L^{\lambda\mu\nu} = g^{\lambda\mu}A^{\nu} - g^{\lambda\nu}A^{\mu} \,. \tag{23}$$

Now our energy momentum tensor is symmetric and we therefore would expect

$L^{\lambda\mu\nu}{}_{;\lambda} = 0$, but instead we find

$$L^{\lambda;\nu}{}_{;\lambda} = A^{\mu;\nu} - A^{\nu;\mu} = F^{\mu\nu} . \tag{24}$$

Thus the electromagnetic field would destroy the conservation of angular momentum of the gravitational field, carrying it off as vortices in the co-ordinate system. To avoid this, what we need is the curl-free part of A^μ. Denoting this by X^μ, we have

$$X_\mu = \nabla_\mu X , \tag{25}$$

where X is a scalar field to be specified later. Now in terms of X_μ, $L^{\lambda\mu\nu}$ may be written

$$L^{\lambda\mu\nu} = (g^{\lambda\mu}g^{\nu\varrho} - g^{\lambda\nu}g^{\mu\varrho})X_\varrho . \tag{26}$$

The expression in the parentheses is anti-symmetric now in both $\mu\nu$ and $\lambda\varrho$ and symmetric under the exchange of $\mu\nu \to \lambda\varrho$, we recognize it as the expression for the Riemann curvature tensor in a space of constant curvature, hence in the absence of matter, we may alternatively write

$$L^{\lambda\mu\nu} = (3/\Lambda) R^{\lambda\varrho\,\mu\nu} X_\varrho . \tag{27}$$

However if we introduce such physical co-ordinates there can be no question of finding co-ordinate transformations connecting these physical co-ordinates with mathematical co-ordinates (unless of course space were flat) since we should have $\mathrm{d}X^\mu = (\partial X^\mu/\partial x^\nu)\,\mathrm{d}x^\nu$ and this is not covariant. It is basically for this reason that people who have tried to find physical co-ordinates by imposing co-ordinate restrictions have encountered serious difficulties. Assuming one found such physical co-ordinates by this method, there would exist co-ordinate transformations connecting them with all mathematical co-ordinate systems in the space by such a relation as above and this is impossible. Since the actual covariant connection between physical co-ordinates and mathematical co-ordinates is of the form

$$DX^\mu = (\partial X^\mu/\partial x^\nu)\,\mathrm{d}x^\nu + \Gamma^\mu_{\lambda\nu} X^\lambda \,\mathrm{d}x^\nu ,$$

it is clear that they may only be related *locally*. Thus in introducing such physical co-ordinates, we are *not* picking out a privileged co-ordinate frame, this would amount to imposing coordinate conditions.

In order to arrive at a way of specifying X_μ such that in the case of special relativity they will go over into the usual Lorentz co-ordinates (for a rec-

tangular co-ordinate system) it is convenient to consider the usual gauge condition in flat space

$$\Box X = 0 \,. \tag{28}$$

The co-ordinate-like solutions to this equation are of the form

$$X = \tfrac{1}{2}(x^0 x^0 + \tfrac{1}{3} x^i x^i) \,, \tag{29}$$

which, however, is not Lorentz invariant. In order to obtain Lorentz invariant co-ordinate-like solutions we must impose the condition

$$\Box X = \delta^\mu_\mu = 4 \tag{30}$$

corresponding to the solution

$$X = \tfrac{1}{2}(x^0 x^0 - x^i x^i) \tag{31}$$

and hence with $X_\mu = \nabla_\mu X = \eta_{\mu\nu} x^\nu$. However this still does not appear quite correct since δ^μ_μ is the trace of the gravitational energy density and we should expect the matter energy trace to influence the co-ordinates, hence we write

$$\Box X = 4 + T = \Lambda^{-1} R \,, \qquad (\varkappa^{-1}\Lambda = 1) \,, \tag{32}$$

where $R = g_{\mu\nu} R^{\mu\nu}$ is the scalar curvature. We see that upon multiplying through by Λ and passing to the limit $\Lambda \to 0$, $R \to 0$, the gauge vanishes. We also obtain the result that for a purely electromagnetic contribution since $T = 0$ the condition reduces to $\Box X = 4$.

Let us now obtain the equations satisfied by the physical co-ordinates themselves. Since the gauge condition may be written $X^\mu{}_{;\mu} = 4 + T$ we have (using the covariant differentiation commution rules and contracting) upon taking ∇_λ of the above expression

$$X_\lambda{}^{;\mu}{}_{;\mu} + R_{\lambda\alpha} X^\alpha = \frac{\partial T}{\partial x^\lambda} \,. \tag{33}$$

If T vanishes, $R_{\lambda\alpha} = \Lambda g_{\lambda\alpha}$ and the equation becomes

$$X_\lambda{}^{;\mu}{}_{;\mu} + \Lambda X_\lambda = 0 \,, \tag{34}$$

Clearly the general solution to the equation consists of « co-ordinates » (which may be thought of as Lorentz co-ordinates plus gravitational perturbations plus « waves »). In the case of the de Sitter metric the co-ordinate

solutions to (34) are readily obtained and will be published later [14]. Let us now turn to the case $R_{\lambda\alpha} = 0$, $\partial T/\partial x^\lambda = 0$, that is assuming that the cosmological constant vanishes and we are outside of matter. Then the equations become

$$X_\lambda^{\ ;\mu}{}_{;\mu} = 0 \,. \tag{35}$$

This bears a resemblance to the harmonic co-ordinate conditions discussed in detail by FOCK [15], the important difference being that the equation is fully covariant and does not imply any restriction on the mathematical co-ordinates used in solving the field equations. More generally, outside matter, we have

$$X^\mu{}_{;\mu} = 4 \,.$$

Finally we should like to remark that although taking $X_\mu = \nabla_\mu X$ rigorously insures a covariant angular momentum conservation law for the gravitational field, for the matter field we find

$$L^{\lambda\mu\nu}{}_{;\lambda} = T^{\lambda\mu} X^\nu{}_{;\lambda} - T^{\lambda\nu} X^\mu{}_{;\lambda} \tag{36}$$

and hence the divergence only vanishes locally since we can then set $X^\nu{}_{;\lambda} = \delta^\nu_\lambda$ and use the symmetry of $T^{\mu\nu}$. Thus we see that in attempting to carry the covariant formalism to its logical conclusion we encounter a serious obstacle in insuring the vanishing of $L^{\lambda\mu\nu}{}_{;\lambda}$ for matter. We may now go back to our original assumption that X_ϱ was vortex-free (conservation of gravitational angular momentum separately) and raise the question as to whether or not one should merely try to insure the conservation of matter + gravitation angular momentum together (in contrast with the separate conservation laws for energy: $g_{\mu\nu;\mu} = 0$, $T_{\mu\nu;\mu} = 0$). In this case the physical co-ordinates could not be vortex free and hence $X_\mu \neq \nabla_\mu X$. However, whether or not $X^{\mu;\nu} - X^{\nu;\mu}$ could be related in a satisfactory way to the electromagnetic field (which would then have the role of providing for a covariant angular momentum conservation law) is an open question.

* * *

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[14] We state the result:

$$X_\mu = \nabla_\mu X \,, \qquad X = \frac{Rt}{3} + \frac{2}{3} R^2 \ln\left(1 - \frac{r^2}{R^2}\right) + \frac{R^2}{9}\left[\left(1 - \frac{r^2}{R^2}\right)^{-\frac{3}{2}} \exp\left(-\frac{3t}{R}\right) - 1\right].$$

Note that in the limit $R \to \infty$ ($\Lambda \to 0$), X tends to the expression (31) upon setting $r^2 = x^i x^i$; also, in polar co-ordinates, $X_\theta = X_\varphi = 0$.

[15] V. FOCK: *The Theory of Space Time and Gravitation* (London, 1959).

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RIASSUNTO

Si esamina l'ipotesi che il termine cosmologico di Einstein (con $\Lambda > 0$) debba essere interpretato come il tensore energia-impulso del campo di gravitazione. Una nuova interpretazione dell'elemento lineare è data dal punto di vista di Mach che conduce in maniera naturale alla necessità di questo termine nelle equazioni del campo. Si considera anche il problema di definire un tensore dell'impulso angolare per il campo di gravitazione.

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Low Energy Limit of Compton Scattering without T, C, P Invariance (*).

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Summary. — When TCP invariance only is valid, Compton scattering from a spin $-\frac{1}{2}$ particle is shown to be entirely determined by its mass, charge, magnetic and electric dipole moment in the two lowest orders of the frequency.

1. – Introduction.

Low energy Compton scattering from spin $-\frac{1}{2}$ particles has been studied by different methods ([1-4]), which assumes separate T, C, P invariance, and was shown to be expressible in terms of the charge, mass and magnetic moment in the first two orders of the photon energy. In ref. ([1]) it is further shown that to this order the classical and the field theoretical calculations coincide after a suitable interpretation. Since P, C invariance must be approximate in Compton scattering, as it may proceed through virtual processes which violate both, and since T invariance may not be exact we shall re-examine this problem by assuming only TCP invariance. It is known that a spin $-\frac{1}{2}$ particle cannot have an electric dipole moment if P or T invariance is valid, whereas if TCP invariance alone should hold, this moment may be different from zero. Thus, in order to compare the field theoretical calculation with the

(*) Supported in part by the United States Atomic Energy Commission Contract No. AT(30-1)-2399.

([1]) M. Gell-Mann and M. L. Goldberger: *Phys. Rev.*, **96**, 1433 (1954).

([2]) F. E. Low: *Phys. Rev.*, **96**, 1428 (1954).

([3]) A. Klein: *Phys. Rev.*, **99**, 998 (1955).

([4]) E. Kazes: *Nuovo Cimento*, **13**, 1226 (1969). Refered to as (K).

classical, to the Lagrangian used by KRAMER, GELL-MANN and GOLDBERGER, an intrinsic electric dipole moment term has been added.

Throughout this work it will be assumed that all interactions are Lorentz invariant and renormalizable. The feasibility of the latter will not be discussed (5).

2. – General considerations.

Let the initial and final photon and fermion momenta be q, p and q', p' respectively. The A amplitude which corresponds to Feynman diagrams connected by a physical Fermion propagator and the B amplitude which yields all other diagrams as given in (K) are

$$(1)\qquad \mathcal{M}^{A}_{\nu\mu}(p', q'; p, q) = \Lambda_\nu(p', p+q)\, S'(p+q)\Lambda_\mu(p+q, p) + \Lambda_\mu(p', p-q')\cdot \cdot S'(p-q')\,\Lambda_\nu(p-q', p)\,,$$

$$(2)\qquad \mathcal{M}^{B}_{\nu\mu}(p', q'; p, q) = -e\left[\left(\frac{\partial}{\partial p'_\mu}+\frac{\partial}{\partial p_\mu}\right)\Lambda_\nu(p', p) + \right.$$
$$\left. + q_\lambda \frac{\partial}{\partial p'_\lambda}\left(\frac{\partial}{\partial p'_\nu}+\frac{\partial}{\partial p_\nu}\right)\Lambda_\mu(p', p) - q'_\lambda \frac{\partial}{\partial p'_\lambda}\left(\frac{\partial}{\partial p'_\mu}+\frac{\partial}{\partial p_\mu}\right)\Lambda_\nu(p', p)\right|_{p'=p},$$

where S' and Λ_μ are the renormalized fermion propagator and vertex operators. As a consequence of Lorentz invariance

$$(3)\qquad S'(p)^{-1} = i\gamma\cdot p\big(G^{(1)}(p^2)+\gamma_5 G^{(2)}(p^2)\big) + M\big(F^{(1)}(p^2)+i\gamma_5 F^{(2)}(p^2)\big)\,,$$

and renormalizability requires that

$$(4)\qquad G^{(1)}(-M^2) = F^{(1)}(-M^2) = 1\,, \qquad G^{(2)}(-M^2) = F^{(2)}(-M^2) = 0\,.$$

Letting

$$(5)\qquad \Lambda_\mu(p', p) = \Lambda^{(1)}_\mu(p', p) + \Lambda^{(2)}_\mu(p', p)\gamma_5\,,$$

$$(6)\qquad P_\alpha = \left(\frac{p'+p}{2}\right)_\alpha, \quad Q_\alpha = (p'-p)_\alpha\,,$$

(5) For a discussion of renormalizability when CP invariance alone is valid see B. D'ESPAGNAT and J. PRENTKI: *Nuovo Cimento*, **6**, 1129 (1957); K. SEKINE: *Nuovo Cimento*: **11**, 87 (1959); C. H. ALBRIGHT, R. HAAG and S. TREIMAN: *Nuovo Cimento*, **13**, 1282 (1955).

it follows that, the most general vertex operator is

$$\Lambda_\mu^{(i)}(p', p) = P_\mu l_{0,1}^{(i)} + Q_\mu l_{0,2}^{(i)} + \gamma_\mu l_{1,1}^{(i)} + P_\mu P\cdot\gamma\, l_{1,2}^{(i)} + \tag{7}$$
$$+ P_\mu Q\cdot\gamma\, l_{1,3}^{(i)} + Q_\mu P\cdot\gamma\, l_{1,4}^{(i)} + Q_\mu Q\cdot\gamma\, l_{1,5}^{(i)} + \sigma_{\mu\nu}P_\nu\, l_{2,1}^{(i)} + \sigma_{\mu\nu}Q_\nu\, l_{2,2}^{(i)} +$$
$$+ P_\mu Q_\lambda\sigma_{\lambda\varrho}P_\varrho\, l_{2,3}^{(i)} + Q_\mu Q_\lambda\sigma_{\lambda\varrho}P_\varrho\, l_{2,4}^{(i)} + \varepsilon_{\mu\alpha\beta\lambda}P_\alpha Q_\beta\gamma_\lambda\gamma_5\, l_{3,1}^{(i)}\,,$$

where $l_{0,1}^{(i)} \equiv l_{0,1}^{(i)}(p'^2, p^2, q^2)$, etc.

The generalized Ward identity now becomes

$$(p'_\mu - p_\mu)\Lambda_\mu^{(i)}(p', p) = e[S'^{(i)}(p')^{-1} - S'^{(i)}(p)^{-1}]\,, \tag{8}$$

where

$$\begin{cases} S'^{(1)}(p)^{-1} = i\gamma\cdot p\, G^{(1)}(p^2) + M\, F^{(1)}(p^2)\,, \\ S'^{(2)}(p)^{-1} = i\gamma\cdot p\, G^{(2)}(p^2) + iM\, F^{(2)}(p^2)\,. \end{cases} \tag{9}$$

Further consequence of eq. (7), (8), (9), are given in the Appendix A. Requiring the renormalized vertex operator to reduce to $ie\gamma_\mu$ for zero momentum transfer when the fermion is on the mass shell and $ip\cdot\gamma = -M$ it follows that

$$l_{1,1}^{(1)} = ie\,, \qquad l_{1,1}^{(2)} = 0\,, \tag{10}$$

$$l_{0,1}^{(i)} + 2iM\, l_{1,2}^{(i)} = 0\,, \tag{11}$$

after using $l_{2,1}^{(i)}(p^2, p^2, 0) = 0$ which follows form the generalized Ward identity, (A.1). Eq. (11) for $i = 2$ is ambiguous in that $ip\cdot\gamma$ can be commuted to the right or left of γ_5 before setting it equal to $-M$, but it will be shown at the end of this section that any of these two alternatives yield $l_{1.0}^{(2)} = l_{1.2}^{(2)} = 0$ for $p^2 = -M^2$, and the ambiguity does not matter.

From Appendix A

$$\begin{cases} l_{1,1}^{(i)}(-M^2, -M^2, 0) = ie\, G^{(i)}(-M^2)\,, \\ l_{1,2}^{(i)}(-M^2, -M^2, 0) = \dfrac{ie}{2}\,\dot{G}^{(i)}(-M^2)\,, \\ l_{0,1}^{(1)}(-M^2, -M^2, 0) = eM\dot{F}^{(1)}(-M^2)\,, \\ l_{0,1}^{(2)}(-M^2, -M^2, 0) = ieM\dot{F}^{(2)}(-M^2)\,. \end{cases} \tag{12}$$

From eq. (11) and (12) for $p^2 = -M^2$ it follows that

$$\begin{cases} \dot{G}^{(1)} = \dot{F}^{(1)}\,, \\ \dot{G}^{(2)} = i\dot{F}^{(2)}\,. \end{cases} \tag{13}$$

Further requirements ont $\dot{G}^{(2)}$ are obtained by performing a spectral decomposition of the propagator; as a consequence of Lorentz invariance it follows that [6]

$$(14)\qquad \langle 0\,|\,\psi_\alpha(x)\overline{\psi}_\beta(y)\,|\,0\rangle = \int \theta(p^0)\,\delta(p^2+m^2)\,\{(ip\cdot\gamma-\sqrt{m^2})\,\varrho_1(m^2)+\varrho_2(m^2)+ \\ + ip\cdot\gamma\gamma_5\varrho_1'(m^2)+i\gamma_5\varrho_2'(m^2)\}_{\alpha\beta}\exp\,[ip\cdot(x-y)\,\mathrm{d}p\,\mathrm{d}m^2\,,$$

where $\psi(x)$ is the renormalized field operator and ϱ_1, ϱ_2, ϱ_1', ϱ_2' are real. Now invoking TCP invariance for the first time [7], it follows that there must be and antiunitary operator U such that

$$(15)\qquad \begin{cases} U\ \psi(x)U^{-1}=(\gamma_5\psi(-x))^+\,, \\ U\psi^+(x)U^{-1}=\gamma_5\psi(-x)\,. \end{cases}$$

Using eq. (14), and (15) it follows that

$$S'(p)=\frac{1}{2\pi}\int\frac{\mathrm{d}m^2}{p^2+m^2-i\varepsilon}\{(ip\cdot\gamma-\sqrt{m^2})\,\varrho_1(m^2)+\varrho_2(m^2)+ \\ + ip\cdot\gamma\gamma_5\,\varrho_1'(m^2)+i\gamma_5\,\varrho_2'(m^2)\}\,.$$

Assuming the photons to have a finite mass λ, and letting M be the lowest fermion mass to enter the spectral decomposition it follows that

$$S'(p)=\frac{1}{ip\cdot\gamma+M}+\frac{1}{2\pi}\int\limits_{(M+\lambda)^2}^{\infty}\frac{\mathrm{d}m^2}{p^2+m^2-i\varepsilon}\,\{(ip\cdot\gamma-\sqrt{m^2})\,\varrho_1(m^2)+ \\ + \varrho_2(m^2)+ip\cdot\gamma\gamma_5\varrho_1'(m^2)+i\gamma_5\varrho_2'(m^2)\}\,.$$

This shows that in the neighborhood of $p^2=-M^2$, $G^{(2)}(p^2)/F^{(2)}(p^2)$ is a real function, further as a consequence of eq. (4) it follows that $\dot{G}^{(2)}(-M^2)/\dot{F}^{(2)}(-M^2)$ is real, but by comparing with the requirement imposed the renormalizability of the vertex function, eq. (13), et follows that $\dot{G}^{(2)}(-M^2)=\dot{F}^{(2)}(-M^2)=0$.

Finally by placing the fermion in a slowly varying magnetic and electric field it follows that its anomalous magnetic moment and electric dipole moment are given by

$$\mu_A = i(l_{2,2}^{(1)}+2M\,l_{3,1}^{(1)})\,,$$
$$d\ = l_{2,2}^{(2)}-2M\,l_{1,3}^{(2)}+4M^2\,l_{2,3}^{(2)}\,,$$

respectively.

(6) H. Lehmann: *Nuovo Cimento*, **11**, 342 (1954).

(7) G. Grawert, G. Lüders and H. Rollnik: *Fortschritte der Physik*, **7**, 291 (1959).

3. – Approximations.

To calculate the A amplitude to the first order in the frequency it necessary to calculate $S'(p+q)$, $\varepsilon_\mu \Lambda_\mu(p+q, p)$ and $\varepsilon'_\nu \Lambda_\nu(p', p'+q')$ up to the first order in q. For the purpose of this calculation after using eq. (13) and the results of Appendix A it follows that

$$(16)\quad S'(p+q) = -\frac{1}{2p\cdot q}\{i(p+q)\cdot\gamma - M + 2p\cdot q(ip\cdot\gamma - M)\dot{G}^{(1)} - 2ip\cdot q q\cdot\gamma \dot{G}^{(1)}\},$$

$$(17)\quad \varepsilon_\mu \Lambda_\mu(p+q, p) = ie\gamma\cdot\varepsilon + 2p\cdot q\gamma\cdot\varepsilon\left(\frac{i\varepsilon}{2}\dot{G}^{(1)} - l^{(1)}_{1,3} - l^{(2)}_{1,3}\gamma_5\right) -$$
$$- 4\varepsilon_\mu\sigma_{\mu\nu}p_\nu p\cdot q\,(l^{(1)}_{2,3} + l^{(2)}_{2,3}\gamma_5) + \varepsilon_\mu\sigma_{\mu\nu}q_\nu(l^{(1)}_{2,2} + l^{(2)}_{2,2}\gamma_5) +$$
$$+ 2\varepsilon_\mu\varepsilon_{\mu\varkappa\lambda\varrho}p_\alpha q_\lambda\gamma_\varrho\gamma_5(l^{(1)}_{3,1} + l^{(2)}_{3,1}\gamma_5)\,,$$

$$(18)\quad \varepsilon'_\nu \Lambda_\nu(p', p'+q') = ie\gamma\cdot\varepsilon' + 2iep'\cdot q'\gamma\cdot\varepsilon'\dot{G}^{(1)} -$$
$$- 2p'\cdot q'\gamma\cdot\varepsilon'\left(\frac{ie}{2}\dot{G}^{(1)} - l^{(1)}_{1,3} - l^{(2)}_{1,3}\gamma_5\right) + 4p'\cdot q'\varepsilon'_\nu\sigma_{\nu\varrho}p'_\varrho(l^{(1)}_{2,3} + l^{(2)}_{2,3}\gamma_5) -$$
$$- \varepsilon'_\nu\sigma_{\nu\varrho}q'_\varrho(l^{(1)}_{2,2} + l^{(2)}_{2,2}\gamma_5) - 2\varepsilon'_\nu\varepsilon_{\nu\varkappa\lambda\varrho}p'_\alpha q'_\lambda\gamma_\varrho\gamma_5(l^{(1)}_{3,1} + l^{(2)}_{3,1}\gamma_5)\,,$$

where all the l's are evaluated at $p'^2 = p^2 = -M^2$, $q^2 = 0$.

The contribution of the B amplitude that results from eq. (2) is

$$\varepsilon'_\nu \mathcal{M}^B_{\nu\mu}\varepsilon_\mu = 4eq_0(2M\,l^{(1)}_{2,3} - l^{(1)}_{3,1})\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon} + 2e\,l^{(2)}_{3,1}(\boldsymbol{q}+\boldsymbol{q}')\cdot(\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon}) +$$
$$+ 2ei(l^{(2)}_{1,3} - 2M\,l^{(2)}_{2,3})\{\boldsymbol{\sigma}\cdot(\boldsymbol{q}+\boldsymbol{q}')\times(\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon}) - \boldsymbol{\sigma}\cdot(\boldsymbol{q}'-\boldsymbol{q})\,\boldsymbol{\epsilon}'\cdot\boldsymbol{\epsilon}\}.$$

Using eq. (1), (16), (17) to obtain the A amplitude and the B amplitude given above, the total scattering amplitude becomes

$$(19)\quad \varepsilon'_\nu \mathcal{M}_{\nu\mu}\varepsilon_\mu = -\frac{e^2}{M}\boldsymbol{\epsilon}'\cdot\boldsymbol{\epsilon} - 2i\frac{\mu^2}{q_0}\boldsymbol{\sigma}\cdot\{(\boldsymbol{\epsilon}'\times\boldsymbol{q}')\times(\boldsymbol{\epsilon}\times\boldsymbol{q})\} +$$
$$+ \frac{2ie}{M}\left(\mu - \frac{e}{4M}\right)q_0\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon} + \frac{ie}{M}\frac{\mu}{q_0}\{\boldsymbol{q}\cdot\boldsymbol{\epsilon}'\,\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}\times\boldsymbol{q}) - \boldsymbol{q}'\cdot\boldsymbol{\epsilon}\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}\times\boldsymbol{q}')\} -$$
$$- 2iq_0 d^2\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon}) - 2i\mu d\boldsymbol{\sigma}\cdot\{(\boldsymbol{q}+\boldsymbol{q}')\times(\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon})\} + 2id\left(\mu + \frac{e}{2M}\right)\boldsymbol{\epsilon}'\cdot\boldsymbol{\epsilon}\boldsymbol{\sigma}\cdot(\boldsymbol{q}'-\boldsymbol{q}),$$

where $\mu = \mu_A + (e/2M)$, and the initial fermion is at rest.

4. – Classical calculation.

Adding an electric dipole moment term to the Lagrangian used in ref. [1] we obtain

$$L = \frac{1}{2} MV^2 + \frac{1}{8\pi}\int (E^2 - H^2)\,\mathrm{d}x + e\boldsymbol{V}\cdot\boldsymbol{A} + \frac{e}{M}\,\boldsymbol{S}\cdot\left[\boldsymbol{H} - \frac{1}{2}\,\boldsymbol{V}\times\boldsymbol{E}\right] +$$
$$+ g_A \boldsymbol{S}\cdot[\boldsymbol{H} - \boldsymbol{V}\times\boldsymbol{E}] + h\boldsymbol{S}\cdot[\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{H}]\,.$$

From this we obtain the following equations of motion

$$M\dot{\boldsymbol{V}} = e\boldsymbol{E} + g_A(\boldsymbol{S}\cdot\nabla)\boldsymbol{H} + \frac{e}{2M}\{\boldsymbol{S}\cdot\nabla)\boldsymbol{H} + \nabla(\boldsymbol{S}\cdot\boldsymbol{H})\} + h(\boldsymbol{S}\cdot\nabla)\boldsymbol{E}\,, \tag{20}$$

$$\nabla\times\boldsymbol{H} - \frac{\partial}{\partial t}\,\boldsymbol{E} = 4\pi\boldsymbol{V}\,\delta(\boldsymbol{R} - \boldsymbol{r}) + 4\pi g\,\nabla\times\delta(\boldsymbol{R} - \boldsymbol{r})\boldsymbol{S} + \tag{21}$$

$$+ 4\pi\left(g_A + \frac{e}{2M}\right)\frac{\partial}{\partial t}\{\delta(\boldsymbol{R} - \boldsymbol{r})\boldsymbol{V}\times\boldsymbol{S}\} + 4\pi h\,\frac{\partial}{\partial t}\{\delta(\boldsymbol{R} - \boldsymbol{r})\boldsymbol{S}\} - 4\pi h\,\nabla\times\{\delta(\boldsymbol{R} - \boldsymbol{r})\boldsymbol{S}\times\boldsymbol{V}\},$$

$$\mathrm{d}\boldsymbol{S}/\mathrm{d}t = g\boldsymbol{S}\times\boldsymbol{H} + h\boldsymbol{S}\times\boldsymbol{E}\,,$$

where on the right hand side of these equations only terms of first order in the fields have been kept, and $g = g_A + (e/M)\cdot\boldsymbol{R}$ is the position of the particle. At large distances from the scattering center the electric field is

$$\boldsymbol{E} = \boldsymbol{E}_0(\boldsymbol{r}, t) + \boldsymbol{E}_{\mathrm{sc}}(\boldsymbol{r}, t)\,,$$

where

$$\boldsymbol{E}_{\mathrm{sc}} = -\frac{1}{r}\left[e\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\boldsymbol{R} - g\left(\hat{r}\times\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\boldsymbol{S} + \hat{r}\cdot\frac{\mathrm{d}^3}{\mathrm{d}t^3}\,\boldsymbol{R}\,\hat{r}\times\boldsymbol{S}\right) + \right. \tag{22}$$
$$+ \left(g_A + \frac{e}{2M}\right)\frac{\mathrm{d}^3}{\mathrm{d}t^3}\,\boldsymbol{R}\times\boldsymbol{S} + h\left(\frac{\mathrm{d}^2\boldsymbol{S}}{\mathrm{d}t^2} + \hat{r}\cdot\frac{\mathrm{d}^3\boldsymbol{R}}{\mathrm{d}t^3}\,\boldsymbol{S}\right) +$$
$$\left. + h\left\{\hat{r}\times\left(\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t}\times\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\boldsymbol{S}\right) - \hat{r}\times\left(\boldsymbol{S}\times\frac{\mathrm{d}^3}{\mathrm{d}t^3}\,\boldsymbol{R}\right)\right\}\right].$$

From eq. (22) it follows that the classical scattering amplitude is

$$f = -\frac{e^2}{M}\,\boldsymbol{\epsilon}'\cdot\boldsymbol{\epsilon} - i\,\frac{g^2}{q_0}\,\boldsymbol{S}\cdot\{(\boldsymbol{\epsilon}'\times\boldsymbol{q}')\times(\boldsymbol{\epsilon}\times\boldsymbol{q})\} + i\,\frac{e}{M}\left(2g_A + \frac{e}{M}\right)q_0\boldsymbol{S}\cdot\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon} + \tag{23}$$
$$+ i\,\frac{eg}{q_0 M}\{\boldsymbol{S}\cdot(\boldsymbol{\epsilon}\times\boldsymbol{q})\boldsymbol{q}\cdot\boldsymbol{\epsilon}' - \boldsymbol{S}\cdot(\boldsymbol{\epsilon}'\times\boldsymbol{q}')\boldsymbol{q}'\cdot\boldsymbol{\epsilon}\} - iq_0\cdot h^2\boldsymbol{S}\cdot(\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon}) -$$
$$- ghi\boldsymbol{S}\cdot\{(\boldsymbol{q} + \boldsymbol{q}')\times(\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon})\} + ih\left(g + \frac{e}{M}\right)\boldsymbol{\epsilon}'\cdot\boldsymbol{\epsilon}\boldsymbol{S}\cdot(\boldsymbol{q}' - \boldsymbol{q})\,.$$

Again it is observed that the classical scattering amplitude becomes identical with the field theoretical calculation under the substitution $S \to \sigma/2$, $g \to 2\mu$, $g_A \to 2\mu_A$, $h \to 2d$.

5. – Discussion.

The terms independent of the electric dipole moment in eq. (19) result from various multipole absorption and emission processes which were discussed in ref. ([1]).

The term proportional to d^2 results from the torque which the electric field exerts on the particle and to it subsequent radiation, thus is caused by $E1$ absorption and re-emission. Whereas $M1$ absorption with $E1$ re-emmission and vice-versa is proportional to

$$E1M1 = \boldsymbol{\epsilon}' \cdot \{\boldsymbol{\sigma} \times (\boldsymbol{\epsilon} \times \boldsymbol{q})\} - \boldsymbol{\epsilon} \cdot \{\boldsymbol{\sigma} \times (\boldsymbol{\epsilon}' \times \boldsymbol{q}')\} = \\ = \boldsymbol{\sigma} \cdot \{(\boldsymbol{q} + \boldsymbol{q}') \times (\boldsymbol{\epsilon}' \times \boldsymbol{\epsilon})\} - \boldsymbol{\sigma} \cdot (\boldsymbol{q}' - \boldsymbol{q})\, \boldsymbol{\epsilon}' \cdot \boldsymbol{\epsilon} \,.$$

This term results from the torque exerted by the external magnetic field on the magnetic moment of the particle and the precession of the electric dipole moment of the particle. $E2$ absorption leads to the acceleration of the particle due to the gradient of the electric field at the particle and will lead to $E1$ radiation

$$E1E2 = \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}\, \boldsymbol{k} \cdot \boldsymbol{\epsilon}' + \boldsymbol{\epsilon}' \cdot \boldsymbol{\epsilon}\, \boldsymbol{\sigma} \cdot \boldsymbol{k} - (\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}'\, \boldsymbol{k}' \cdot \boldsymbol{\epsilon} + \boldsymbol{\epsilon}' \cdot \boldsymbol{\epsilon} \boldsymbol{\sigma} \cdot \boldsymbol{k}') = \\ = -\sigma \cdot \{(\boldsymbol{q} + \boldsymbol{q}') \times (\boldsymbol{\epsilon}' \times \boldsymbol{\epsilon})\} - \boldsymbol{\sigma} \cdot (\boldsymbol{q}' - \boldsymbol{q})\, \boldsymbol{\epsilon}' \cdot \boldsymbol{\epsilon} \,.$$

It thus follows that the term proportional to $\mathrm{d}\mu$ in eq. (19) is due $E1M1 - E1E2$ and that proportional to $\mathrm{d}(\mu + (e/2M))$ is due $E1M1 + E1E2$ to transitions.

It is clear that if P invariance is valid, the second term of eq. (5) vanishes making $d = 0$ and yields the same results obtained previously with T, C, P invariance ([1-4]). If T invariance alone should be valid

$$\Lambda^T_\mu(\boldsymbol{p}', p_0'; \boldsymbol{p}, p_0) = \pm \gamma_2 \Lambda_\mu(\boldsymbol{p}, -p_0; \boldsymbol{p}', -p_0')\gamma_2 \,,$$

where the upper sign is used when $\mu = 1, 2, 3$ ([8]). Since $\gamma_2 \gamma_\mu \gamma_2 = \pm \gamma_\mu^T$ it follows that $l_{1,3} = l_{2,2} = l_{2,3} = 0$ for $p'^2 = p^2 = -M^2$ and $q^2 = 0$, thus $d = 0$. Of course these results are expected on even more general grounds.

([8]) The corresponding discussion in (K) is incorrect.

* * *

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APPENDIX

From eq. (8) it follows that:

(A.1) $$l^{(i)}_{2,1} + P\cdot Q\, l^{(i)}_{2,3} + Q^2 l^{(i)}_{2,4} = 0\,,$$

(A.2) $$P\cdot Q\, l^{(1)}_{0,1} + Q^2 l^{(1)}_{0,2} = eM(F^{(1)}(p'^2) - F^{(1)}(p^2))\,,$$

(A.3) $$P\cdot Q^{(2)}_{0,1} + Q^2 l^{(2)}_{0,2} = eMi(F^{(2)}(p'^2) - F^{(2)}(p^2))\,,$$

(A.4) $$l^{(i)}_{1,1} + P\cdot Q(l^{(i)}_{1,2} + l^{(i)}_{1,3}) + Q^2(l^{(i)}_{1,4} + l^{(i)}_{1,5}) = ieG^{(i)}(p'^2)\,,$$

(A.5) $$l^{(i)}_{1,1} - P\cdot Q(l^{(i)}_{1,2} - l^{(i)}_{1,3}) - Q^2(l^{(i)}_{1,4} - l^{(i)}_{1,5}) = ieG^{(i)}(p^2)\,.$$

In the Appendix of (K) it is incorrectly stated that eq. (A.7), (A.8), (A.9), (A.10) need be valid for the consistency of the calculations of the B diagrams. In fact the generalized Ward identity suffices. Letting

$$\frac{\mathrm{d}}{\mathrm{d}p'^2}\, l \equiv \overset{1}{l}\,, \qquad \frac{\mathrm{d}}{\mathrm{d}p^2}\, l \equiv \overset{2}{l}\,,$$

from eq. (A.1) it follows that for $q^2 = 0$,

$$\overset{1}{l}{}^{(i)}_{2,1} + l^{(i)}_{2,3} = 0\,, \qquad \overset{2}{l}{}^{(i)}_{2,1} - l^{(i)}_{2,3} = 0\,,$$

and from eq. (A.4), (A.5) the following are readily established

$$\overset{1}{l}{}^{(i)}_{1,1} + l^{(i)}_{1,3} = \frac{ie}{2}\dot{G}^{(i)}\,, \qquad \overset{2}{l}{}^{(i)}_{1,1} - l^{(i)}_{1,3} = \frac{ie}{2}\dot{G}^{(i)}\,,$$

when $q^2 = 0$.

RIASSUNTO (*)

Si dimostra che, quando è valida solo l'invarianza TCP, lo scattering Compton da una particella di spin $\frac{1}{2}$ viene completamente determinato dalla sua massa, dalla sua carica e dal suo momento di dipolo magnetico ed elettrico, nei due più bassi ordini della frequenza.

(*) *Traduzione a cura della Redazione.*

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Les invariances de jauge classiques de l'électrodynamique quantique (*).

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Summary. — One first recalls the principles of gauge invariance and studies their violations occuring in perturbation theory: *i.e.*, the well known photon self-energy; the commonly ignored photon-photon interaction term. One then constructs a new equation for the photon field, involving new renormalization constants, which is such that the S-matrix is gauge invariant, without using regularization. Gauge invariance of the field equations supplies relations for the new renormalization constants. Non perturbative calculation proves the consistency of these relations and connects the new constants to the observable quantities. The values of these constants are then calculated to every order of perturbation theory.

Introduction.

Dès la fondation de l'électrodynamique quantique on s'est heurté à des difficultés pour préserver l'invariance de jauge de la théorie, en raison de l'apparition, dans les calculs explicites, d'une masse non nulle du photon. Les méthodes covariantes et la renormalisation qui ont résolu plusieurs des difficultés liées à l'apparition d'infinis dans l'ancienne théorie, ne sont généralement pas appliquées au problème de la masse du photon, en raison de la difficulté de principe de concilier l'existence de l'invariance de jauge avec la présence d'un terme de masse du photon nue dans les équations.

En théorie de perturbation cette difficulté parait liée à l'ambiguité de la définition de produit de distributions, et est généralement résolue en supprimant les singularités des fonctions Δ par les méthodes de régularisation.

(*) Ce travail a bénéficié de l'aide du Commissariat à l'energie atomique.

Mais un élément nouveau dans ce problème est apporté par l'usage des méthodes de calcul global. Elles montrent, en effet, que la self-énergie du photon, n'a plus une valeur ambiguë, mais peut, au contraire, être définie en fonction de quantités en principe observables. Par ailleurs, si on admet à la fois l'exactitude des équations de l'électrodynamique et l'absence d'états de normes négatives dans cette théorie, la masse du photon nue ne peut être nulle, et la difficulté de principe reparait.

Cependant la self-énergie du photon, n'est pas le seul terme de violation de l'invariance de jauge: comme nous le montrerons, en effet, l'interaction photon-photon, calculée en théorie de perturbation, n'est pas non plus invariante de jauge, en dépit d'une opinion répandue. Il n'existe malheureusement pas, pour l'instant, de méthode de calcul global qui soit capable d'exprimer la nouvelle constante d'interaction, en fonction des grandeurs observables, nous assurant ainsi de sa non ambiguité.

Le point de vue que nous admettrons, dans cet article, sera, en attendant que des progrès soient réalisés dans l'étude des méthodes de calcul global, de prendre au sérieux toutes les violations d'invariance de jauge que nous rencontrons en théorie de perturbation.

C'est dans cet esprit que nous allons réexaminer les paradoxes de l'invariance de jauge et en donner une nouvelle solution (*).

Nous rappellerons d'abord à la Section **1** les différents principes d'invariance de jauge que l'on postule pour l'électrodynamique et les propriétés physiques observables et mathématiques qui devraient en résulter.

Nous dresserons ensuite, à la Section **2**, un catalogue des diverses violations de ces propriétés qui apparaissent dans les solutions, en théorie de perturbation, des équations habituellement utilisées, et nous discuterons des difficultés que soulèvent ces violations.

Nous unifierons, à la Section **3**, les divers résultats de la Section précédente dans une formule unique qui relie entre elles diverses formes de violation des propriétés d'invariance de jauge.

A la Section **4**, nous construirons de nouvelles équations pour l'électrodynamique, qui ont précisément, pour conséquences, les propriétés convenables d'invariance de jauge. Exprimant que les équations elles mêmes doivent être invariantes sous le groupe de jauge, nous en déduirons que les nouvelles constantes de renormalisation, intervenant dans les nouvelles équations de l'électrodynamique, doivent satisfaire à certaines relations.

A la Section **5**, nous étudierons avec les méthodes de calcul global, les nouvelles équations de l'électrodynamique, confirmant ainsi la cohérence des

(*) Un résumé des principaux résultats contenus dans cet article a été publié: B. JOUVET: *Compt. Rend. Acad. Sci.*, **251**, 1119 (1960).

relations énoncées à la Section 4, et reliant, en partie, les nouvelles constantes de renormalisation aux grandeurs observables.

Au moyen des relations obtenues aux Sections 4 et 5 nous pourrons alors, aux Sections 6 et 7, calculer les nouvelles constantes de renormalisation à tout ordre de la théorie de perturbation. Les implications de ces résultats, sur la forme que doivent avoir, à tout ordre de la théorie de perturbation, les grandeurs observables ne seront pas abordées dans ce travail.

1. – Rappel des principes d'invariance de jauge et leurs conséquences.

1) L'invariance de jauge a son origine dans *l'électromagnétisme* classique de Maxwell, qui, ne faisant intervenir que les champs $B_{\mu\nu} = \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu$ est invariante pour le changement

$$\varphi_\mu \to \tilde{\varphi}_\mu = \varphi_\mu + \partial_\mu \Phi \,. \tag{1}$$

L'antisymétrie de $B_{\mu\nu}$ implique, de plus, la conservation du courant.

2) *Dans la théorie électrodynamique* non super-quantifiée le lagrangien du système de champ de Dirac et de champ de Maxwell en interaction $\mathscr{L}_1 = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \bar{\psi}(\gamma\hat{c} + m)\psi + ie\bar{\psi}\gamma^\mu \psi q^\mu$ est invariant pour la transformation plus générale associant au changement (1) le changement

$$\psi \to \tilde{\psi} = \psi \exp[ie\Phi] \,. \tag{2}$$

Physiquement, cette invariance signifie que si on plonge un système physique dans un potentiel gradient externe, aucun effet n'en résultera sur le lagrangien, donc sur les équations de mouvement et les éléments de matrice S, si on prend soin de choisir en même temps une nouvelle jauge pour les grandeurs ψ, et inversement. Ainsi les propriétés d'un électron dans un potentiel gradient $\partial_\mu \Phi$ peuvent aussi bien se calculer à partir de celles d'électrons dans le vide, pourvu que l'on choisisse pour les spineurs qui les décrivent une jauge convenable $\tilde{\psi} = \psi \exp[ie\Phi]$. Si Φ est une constante, le groupe (2) laisse aussi invariant séparément le lagrangien du champ de Dirac sans interaction. On démontre à l'aide du principe variationnel, que cette « invariance de phase » suffit pour entrainer la conservation du courant $j_\mu = ie\psi\gamma_\mu\psi$. Mais une autre propriété qui n'est pas reliée simplement à l'invariance mathématique du lagrangien existe aussi dans cette théorie: Grace à la conservation du courant aucun effet observable ne résulte de la présence d'un potentiel gradient sur la diffusion d'un électron. Par ailleurs, un changement de jauge des spineurs est sans effet sur les sections efficaces de diffusion qui sont en $\psi^*\psi$. Ainsi un

seul changement (1) ou (2), et non ((1) + (2)), est sans effet observable sur une diffusion d'un électron, alors même que ce changement ne laisse invariant ni le lagrangien ni les équations du mouvement.

3) *En électrodynamique quantique* on transpose et généralise simplement le principe précédent. On postule en effet que les équations sont invariantes sous le groupe ((1)+(2)). On peut généraliser en supposant que le potentiel Φ est quantifié, ajoutant un lagrangien $\mathscr{L}_q = -\frac{1}{2}\partial_\mu \Phi \partial_\mu \Phi$. Nous appellerons: invariance de phase, invariances de jauge classiques constantes, restreintes et générales, et invariance de jauge quantique

les cinq cas où

$\Phi =$ constante

$\Phi = A_\mu x^\mu$ ($A_\mu =$ constantes)

$\Phi = \Phi(x)$ champ non quantifié tel que $\Box\Phi = 0$

$\Phi = \Phi(x)$ champ non quantifié quelconque

$\Phi = \Phi(x)$ opérateur quantifié.

Nous nous limiterons dans cet article aux seules invariances de jauge classiques.

Il est clair que si les équations du mouvement des champs possèdent certaines invariances, les prédictions sur le mouvement des champs que l'on peut en déduire en résolvant ces équations doivent aussi posséder ces invariances.

L'emploi de la matrice S n'étant qu'une méthode d'intégration de ces équations, les éléments de matrice qui correspondent effectivement aux intégrales des équations du mouvement doivent donc posséder ces invariances. Il en résulte selon le groupe postulé, des propriétés correspondantes « d'invariance pratique de jauge » que nous allons énoncer.

A) *L'invariance de phase.* – On déduit du lagrangien par la méthode variationnelle, ou directement des équations, la relation: $\partial_\mu \bar{\psi}\gamma_\mu\psi = 0$.

B) *L'invariance de jauge classique* [groupe ((1)+(2))].

B-1. Cette invariance signifie que l'on peut calculer les propriétés dynamiques d'un électron dans un potentiel gradient externe, en calculant celle d'un électron en l'absence de ce potentiel, pourvu que l'on ait transformé la jauge de son champ ψ en $\tilde{\psi}\exp[ie\Phi]$. Dans un potential général $\varphi_\mu = \varphi_\mu^{(1)} + \partial_\mu\Phi$ on a, notant $\psi' \equiv \psi(x')$:

$$\langle 0 \,|\, \bar{\psi}\psi' \,|\, 0\rangle_{\varphi_\mu^{(1)}+\partial_\mu\Phi} = \langle 0 \,|\, \tilde{\bar{\psi}}\,\tilde{\psi}' \,|\, 0\rangle_{\varphi_\mu^{(1)}} \exp[ie(\Phi' - \Phi)] \tag{3}$$

et le propagateur

$$S'_F(x, y; \varphi^{(1)}_\mu + \partial_\mu \Phi) = S'_F(x, y; \varphi^{(1)}_\mu) \exp\left[ie\left(\Phi(y) - \Phi(x)\right)\right]. \tag{4}$$

a) Dans le cas où $\varphi^{(1)}_\mu = 0$ et $\Phi = A_\mu x^\mu$ (invariance de jauge constante) on en déduit par transformation de Fourier $S'_F(p; A) = S'_F(p + eA)$ et les relations restreintes de Ward [1] entre opérateurs vertex et propagateur d'électrons, renormalisés,

$$i\Gamma^{(n)}_{\mu\nu\ldots}(p, p) = + \left(\frac{\partial^n S'^{-1}_F(p, A)}{\partial(eA_\mu)\partial(eA_\nu)\ldots} \right)_{A=0} = + \frac{\partial^n S'^{-1}_F(p)}{\partial p_\mu \partial p_\nu \ldots}.$$

Puisque l'électron peut interagir avec le potentiel A_μ, soit directement, soit par l'intermédiaire de son propre nuage de photons, on voit que ce théorème implique que l'interaction d'un photon d'impulsion nulle (A_μ) avec les photons du nuage doit être nulle.

b) Dans le cas de l'invariance de jauge classique générale, utilisons la dérivée variationelle [2] pour calculer l'opérateur $\Gamma_\mu(x, z, y)$ que nous définissons par:

$$\left(\frac{\partial S'_F(x, y; \varphi_\mu)}{\delta(e\varphi_\mu(z))} \right)_{\varphi_\mu = 0} = \int S'_F(x - x') \Gamma_\mu(x', z, y') S'_F(y' - y)\, \mathrm{d}x'\, \mathrm{d}y', \tag{5}$$

développant le premier membre de (5) on a:

$$S'_F(x, y; \partial_\mu q) - S'_F(x - y) = \int \left(\frac{\delta S'_F(x, y; \partial_\mu \varphi)}{\delta(e\partial_\mu \varphi(z))} \right)_{\delta\mu\Phi = 0} \delta(e\partial_\mu \Phi(z))\, \mathrm{d}z =$$

$$= -\int \frac{\partial}{\partial z_\mu} \left(\frac{\delta S'_F(x, y; \partial_\mu \Phi)}{\delta(e\partial_\mu \Phi(z))} \right)_{\delta_\mu \Phi \to 0} e\Phi(z)\, \mathrm{d}z + \text{terme de surface.}$$

Le second membre donne par ailleurs:

$$S'_F(x, y; \partial_\mu \Phi) - S'_F(x - y) = ieS'_F(x - y) \int \left(\delta(z - y) - \delta(z - x)\right) \Phi(z)\, \mathrm{d}z.$$

On en déduit alors la première des relations de Ward généralisées [3]:

$$\int S'_F(x - x') \frac{\partial}{\partial z_\mu} \Gamma_\mu(x', z, y') S'_F(y' - y)\, \mathrm{d}x'\, \mathrm{d}y' =$$

$$= -i \left(S'_F(x - z)\delta(z - y) - S'_F(z - y)\delta(z - x) \right),$$

[1] J. C. Ward: *Phys. Rev.*, **77**, 293 (1950); **78**, 182 (1950).
[2] J. Schwinger: *Proc. Nat. Acad. Sci.*, **37**, 452 (1951).
[3] Y. Takahashi: *Nuovo Cimento*, **7**, 371 (1957); H. S. Green: *Prog. Phys. Soc.*, **66**, 873 (1953).

dont la forme, plus connue dans l'espace des impulsions, est

$$ik^{\mu}\Gamma_{\mu}(p, p+k) = S_F'^{-1}(p+k) - S_F'^{-1}(p) . \tag{6}$$

Les autres relations portant sur les vertex à N photons $\Gamma^{(n)}_{\mu\nu\ldots}$ peuvent aussi s'obtenir en poursuivant la méthode (*) aux ordres supérieurs du développement en Φ.

B-2) Cette invariance a aussi des conséquences sur l'interaction photon-photon: le principe d'invariance de jauge implique en effet que les éléments de matrice, fonction de l'opérateur φ_{μ} et caractérisant les mouvements et interactions des photons dans le potentiel de jauge $\partial_{\mu}\Phi$, doivent être égaux aux éléments de matrice similaires, fonction de l'opérateur $(\varphi_{\mu} - \partial_{\mu}\Phi)$ et caractérisant les mouvements et interactions des photons dans le vide.

Dans le cas du mouvement d'un seul photon, l'élément de matrice convenable, le propagateur d'un photon dans le potentiel de jauge, se déduit de l'expression [2]

$$\begin{aligned} I_2^{\mu\nu} &= \langle 0|\varphi_{\mu}\varphi_{\nu}'|0\rangle_{\Phi} - \langle 0|\varphi_{\mu}|0\rangle_{\Phi}\langle 0|\varphi_{\nu}'|0\rangle_{\Phi} \\ &= \langle 0|(\varphi_{\mu} - \partial_{\mu}\Phi)(\varphi_{\nu}' - \partial_{\nu}'\Phi')|0\rangle - \langle 0|(\varphi_{\mu} - \partial_{\mu}\Phi)|0\rangle\langle 0|(\varphi_{\nu}' - \partial_{\nu}'\Phi')|0\rangle : \end{aligned}$$

or cette dernière expression est, une fois developpée, identique à

$$\langle 0|\varphi_{\mu}\varphi_{\nu}'|0\rangle - \langle 0|\varphi_{\mu}|0\rangle\langle 0|\varphi_{\nu}'|0\rangle$$

et par conséquent indépendante du potentiel de jauge. Il n'y a donc pas d'interaction d'un photon avec les champs de jauge. Ce résultat est plus général que celui qu'impliquent indirectement les relations restreintes de Ward. On généralise ce résultat aux interactions faisant intervenir plus de deux photons, en prenant garde de ne considérer que les termes d'interaction vraie $I^{(n)}_{\mu\nu\ldots}$ entre (n) photons. Ils correspondent aux fonctions $\varrho(x_1 \ldots x_n)$ définies par FREESE [4].

Ainsi jusqu'à l'ordre de 4, on a, en notation symbolique, pour des opérateurs de photon ayant tous le même indice:

$$(\varphi^2)_0 = I_2 + (\varphi)_0(\varphi)_0 \; ; \qquad (\varphi^3)_0 = I_3 + 3I_2\cdot(\varphi)_0 + (\varphi)_0(\varphi)_0(\varphi)_0 \; ;$$

$$(\varphi^4)_0 = I_4 + 4I_3\cdot(\varphi)_0 + 3I_2\cdot(\varphi)_0(\varphi)_0 + 3I_2\cdot I_2 + (\varphi)_0(\varphi)_0(\varphi)_0(\varphi)_0 \; ; \; \ldots$$

(*) Nous tenons à remercier R. STORA qui nous a indiqué cette méthode de démonstration.

(4) E. FREEZE: *Nuovo Cimento*, **2**, 50 (1955).

on en déduit alors l'invariance de I_3 et I_4 de la même façon que nous l'avons montré pour I_2.

Ces théorèmes nécessitent que nous fassions maintenant les distinctions suivantes:

— Nous appelons *invariance théorique de jauge*, l'existence, au moins pour les équations, sinon pour le lagrangien, du groupe d'invariance $((1)+(2))$.

— Nous appelons *invariance pratique de jauge*, les propriétés que nous venons d'énoncer (théorème de Ward, restreint et général et invariance des interactions de photons); cette propriété très particulière des tenseurs d'interaction entre photons d'être invariant sous le seul groupe (1), est généralement aussi appelée invariance de jauge [5], laissant croire que ce seul groupe (1) est un groupe d'invariance de la théorie alors qu'il ne laisse pas invariant les équations du mouvement.

Par ailleurs, une autre confusion se rencontre souvent à propos de la définition de ce que l'on appelle quantité invariante ou non invariante de jauge. Par exemple $\langle 0 | \bar{\psi}\psi' | 0 \rangle$ est invariant sous le groupe $((1)+(2))$, mais ne l'est pas sous le seul groupe (1) ou (2), ou $\langle 0 | \varphi_\mu \varphi'_\nu | 0 \rangle$ n'est invariant ni sous le groupe (1) ou (2) ni sous le groupe $((1)+(2))$.

Pour éviter de telles confusions nous spécifierons dans tous les cas le groupe sous lequel telle quantité est invariante ou non.

Dans la théorie usuelle on postule un lagrangien transposé du lagrangien classique mais faisant intervenir des grandeurs non renormalisées; il est invariant sous un groupe $((1)+(2))$ qui porte sur le quantités non renormalisées. Il semblerait que les théorèmes généraux, que nous venons de rappeler, doivent avoir pour conséquences pratiques que les tenseurs $M^{(n)}_{\mu\nu\varrho\ldots}(k_1 k_2 \ldots k_n)$ representant l'ensemble des boucles fermées, d'où partent (n) photons, soient tels que [6]

$$k_\mu^1 M^{(n)}_{\mu\nu\varrho\ldots} = k_2^\nu M^{(n)}_{\mu\nu\varrho\ldots} = \ldots = 0 ,$$

donc

$$M^{(n)}_{\mu\nu\varrho\ldots} \sim c \cdot k_1^\mu \cdot k_2^\nu \ldots \qquad \text{si} \quad k_i \to 0$$

et en particulier

$$M^{(n)}_{\mu\nu\varrho\ldots}(k_1, k_2, \ldots, k_i = 0, \ldots, k_n) = 0 .$$

Il en résulterait ainsi que le photon doive conserver la masse nulle que lui ont donné les équations du mouvement des champs nus.

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[5] N. N. Bogoliubov et P. V. Shirkov: *Introduction to the Theory of Quantized Field* (traduction par S. M. Volkoff) (New York, 1959), p. 361.

[6] J. M. Jauch et F. Rohrlich: *The Theory of Photons and Electrons* (Cambridge, 1955), p. 160.

Il est généralement donné une démonstration de ces propriétés à partir de l'invariance (?) sous le seul groupe (1) [7], ou bien en calculant $\langle 0 | \overline{\psi}\gamma_\mu\psi | 0\rangle$ dans un champ externe, et en exigeant que $\partial_\mu\langle\overline{\psi}\gamma_\mu\psi|0\rangle = 0$ établissant ainsi une connexion entre ces propriétés et le seul groupe d'invariance de phase, mais non le groupe de jauge ((1)+(2)).

Mais c'est un fait bien connu, que nous allons maintenant préciser, que ces prédictions ne sont pas vérifiées dans les calculs explicites.

2. – Les contradictions au principe d'invariance de jauge en théorie de perturbation.

L'exemple classiquement cité de contradiction aux théorèmes précédent est fourni par le calcul de la masse non nulle du photon; ce n'est pas le seul cas, comme nous le verrons. Pour élucider le plus complètement possible les propriétés et les relations des différents termes parasites, nous allons rappeler et préciser certains résultats généralement connus de la *théorie de perturbation*, et corriger certaines erreurs courantes faites à ce sujet:

a) Le terme parasite de plus bas ordre en e rencontré est l'expression du *courant de vide* $\langle 0 | ie\overline{\psi}\gamma_\mu\psi | 0\rangle$, dont le calcul montre qu'il n'est pas nul mais divergent.

L'antisymétrisation du courant, selon Heisenberg,

$$j_\mu = ie\overline{\psi}\gamma_\mu\psi \to \frac{ie}{2}\,[\overline{\psi}, \gamma_\mu\psi]\,,$$

rend bien j_μ antisymétrique de charge $(j_\mu^c(x) = -\,j_\mu(x))$ mais ne supprime pas la difficulté en dépit de la preuve générale [8] basée sur la symétrie de charge.

L'expression est en effet ambiguë, et il faut faire le passage à la limite [9]

$$j_\mu(x) = \lim_{x'\to x}\frac{ie}{4}\left([\overline{\psi}(x), \gamma_\mu\psi(x')] + [\overline{\psi}(x'), \gamma_\mu\psi(x)]\right),$$

ou utiliser la régularisation pour obtenir le résultat voulu. La connexion du terme de courant du vide avec le problème de l'invariance de jauge apparaît lorsque l'on veut utiliser la méthode générale qui consiste à renormaliser le courant du vide, en soustrayant au lagrangien le contre-terme non invariant $\varphi_\mu\langle j_\mu\rangle_0$.

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[7] G. Källén: *Handb. d. Phys.*, **5**, 281 (1958); J. Schwinger: *Phys. Rev.*, **76**, 815 (1949).

[8] J. M. Jauch et F. Rohrlich: p. 160, réf. [6].

[9] W. Pauli: *Feldquantisierung* (1920) p. 16.

b) Le tenseur de polarisation du vide, du premier degré en α, $K^{(1)}_{\mu\nu}(p^2)$, fournit, outre l'expression invariante pratique de jauge en $(p^2\delta_{\mu\nu} - p_\mu p_\nu)$ divergent logarithmique, le tenseur divergent quadratiquement

$$\frac{\alpha}{2\pi}\left[(\infty - m^2) - \frac{p^2}{3}\right]\delta_{\mu\nu}\,.$$

Le terme constant de ce tenseur, donne au photon une masse (divergente) en contradiction avec les théorèmes énoncés plus haut.

Il est d'usage de sortir de cette difficulté en pratiquant sur la seule expression non invariante de jauge ([10]), l'artifice de calcul de la régularisation ([11]) qui a l'effet cherché de la rendre nulle. Mais alors une autre difficulté demeure liée à la non invariance de jauge. En effet, nous avons scindé

$$K^{(1)}_{\mu\nu}(p^2) = (A + Bp^2)\,\delta_{\mu\nu} + (p^2\delta_{\mu\nu} - p_\mu p_\nu)(c + R(p^2))\,, \qquad R(0) = 0\,,$$

à priori, en deux tenseurs dépendants et invariants de jauge. Il est clair qu'une autre séparation est possible, soit:

$$K^{(1)}_{\mu\nu}(p^2) = \left(A + Bp^2 + p^2(c + R(p^2))\right)\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + (A + Bp^2)\,\frac{p_\mu p_\nu}{p^2}\,;$$

l'application de la régularisation sur le seul tenseur non invariant en $p_\mu p_\nu/p^2$, ne résoud plus alors la difficulté. De plus, la constante B contribue à la valeur de la constante de renormalisation Z_3.

On pourrait supprimer ce terme de masse dans les effets observables par une renormalisation ([12]), introduisant dans le lagrangien un contre terme convenable en $\partial_{\mu''}q_\mu q''$. Mais alors le lagrangien (et les equations) perdent l'invariance théorique qui a été postulée et on ne voit pas pourquoi, il y aurait alors une invariance des éléments observables si $\mathscr{L}$ ne l'est pas?

Certes il est difficile de prouver généralement, qu'une telle situation est impossible, et l'exemple de l'invariance sous le seul groupe (1) pour les tenseurs $I^{\mu\nu}_{(0)}$ peut être une indication que les solutions d'équations peuvent posséder d'autres invariances que celles des équations elles-mêmes.

Néanmoins l'espoir de l'explication de ce paradoxe, par cette voie, a toujours été considéré comme assez mince, au point que l'on préfère admettre l'emploi des méthodes de régularisation. Une autre justification de la régularisation est l'opinion souvent exprimée que les difficultés d'invariance de

([10]) W. Pauli: p. 34, réf. ([9]); G. Källén: p. 281, réf. ([7]).

([11]) W. Pauli et F. Villars: *Rev. Mod. Phys.*, **21**, 434 (1949); J. Schwinger: *Phys. Rev.*, **82**, 664 (1951); W. Guttinger: *Prog. Theor. Phys.*, **13**, 612 (1955).

([12]) J. M. Jauch et F. Rohrlich: p. 190, réf. ([6]).

jauge, coïncidant avec l'apparition d'infinis, pourraient être un des défauts supposés de la théorie de perturbation en représentation d'interaction comme en représentation de Heisenberg.

Mais on se convainc facilement, en utilisant des relations globales du type de Lehmann, que ce paradoxe n'est pas lié à un éventuel défaut de cette méthode ([13]) (*).

La constante finie, B, coefficient de p^2 dans le tenseur en $\delta_{\mu\nu}$ viole aussi l'invariance pratique de jauge. Elle disparait par régularisation, mais peut être soustraite par l'addition à $\mathscr{L}$ d'un contre terme convenable en $((\partial_\mu \varphi'')^2)$, qui viole l'invariance de jauge générale mais non l'invariance restreinte ($\Box \Phi = 0$).

c) Il peut sembler un non sens au lecteur que nous rangions *l'interaction photon-photon* comme un autre cas de violation de l'invariance pratique de jauge de la théorie conventionnelle. En effet, l'opinion est bien établie que, au moins au plus bas ordre de la théorie de perturbation, l'interaction photon-photon est invariante de jauge. L'argument avancé pour le montrer est que la somme des trois graphes divergents logarithmiquement représentant « l'interaction photon-photon étant finie, l'expression doit être invariante de jauge » ([14]).

Cette assertion repose sur l'impression, qu'accrédite le succès de la régularisation dans le problème de la masse du photon, que les violations d'invariance sont necessairement en relation avec l'apparition de divergences propres à la théorie de perturbation. Or comme nous l'avons mentionné à propos de la self-énergie du photon, il n'en est rien. En fait, *l'interaction photon-photon, quoique finie, n'est pas invariante pratique de jauge.* Pour s'en convaincre il suffit de la calculer: Si le tenseur $M^{(4)}_{\mu\nu\varrho\sigma}$ de l'interaction photon-photon était invariante de jauge, il devrait être nul lorsque l'un des photons porte une impulsion nulle. Un tel calcul, très explicite, a été publié par KARPLUS et NEUMAN ([15]).

On voit alors que le tenseur $M^{(4)}_{\mu\nu\varrho\sigma}$ (page 382, formule (15) de ces auteurs), comprend deux groupements de termes: le premier en $(\delta_{\mu\nu}\delta_{\varrho\sigma}+\delta_{\mu\sigma}\delta_{\nu\varrho}-2\delta_{\mu\varrho}\delta_{\nu\sigma})$ est identiquement nul, avec ou sans régularisation par raison de symétrie; par contre le second, en $m^4(\delta_{\mu\nu}\delta_{\varrho\sigma}+\delta_{\mu\sigma}\delta_{\nu\varrho}-\delta_{\mu\varrho}\delta_{\nu\sigma})$, qui s'annule par régularisation, est fini non nul sans régularisation.

Pour annuler à cet ordre, dans la matrice S, l'interaction photon-photon, de zéro énergie-impulsion, il faut donc ajouter au lagrangien, qui n'est plus

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([13]) H. N. FRIED: *Phys. Rev.*, **118**, 1643 (1960); L. EVANS, G. FELDMAN et P. T. MATTHEWS: (prepublication, 1960).

(*) Cet aspect de la question sera discuté en détail à la Sect. 5.

([14]) J. M. JAUCH et F. ROHRLICH: p. 290, réf. ([6]).

([15]) R. KARPLUS et M. NEUMAN: *Phys. Rev.*, **80**, 380 (1950).

invariant théorique de jauge, le contre terme de la forme $\lambda_4(\varphi_\mu\varphi^\mu)^2$. On calcule qu'à cet ordre $\lambda_4 = \alpha^2/3!$.

Il est important de noter que la constante finie qui apparait ici ne peut être due à un déplacement de variables incorrect puisque le graphe est au plus logarithmiquement divergent.

Cependant, ce terme étant soustrait, il reste encore à prouver que le tenseur renormalisé est alors effectivement invariant pratique de jauge, c'est-à-dire bien tel que $k_\mu^1 M^{\text{Ren.}}_{\mu\nu\varrho\sigma}(k_1k_2k_3k_4) = 0$.

Karplus et Neuman ont donné une telle démonstration, mais qui est aussi entachée par l'usage de la régularisation. Sans régularisation, les deux expressions qu'ils considèrent,

$$(7)\qquad \int d^4p \left\{ \frac{\mathrm{Tr}\,((i\gamma p - m)\gamma_\nu(i\gamma(p-k_2)-m)\gamma_\lambda(i\gamma(p-k_2-k_3)-m)\gamma_\sigma)}{D(p)D(p-k_2)D(p-k_2-k_3)} - \right.$$
$$\left. - \frac{\mathrm{Tr}\,((i\gamma p - m)\gamma_\lambda(\gamma i(p+k_3)-m)\gamma_\nu(i\gamma(p+k_2+k_3)-m)\gamma_\sigma)}{D(p)D(p+k_3)D(p+k_2+k_3)} \right\},$$

où
$$D(p) = p^2 + m^2\,,$$

ne se compensent plus, après le déplacement de variables $(p \to p - k_2 - k_3)$ dans la seconde, car elles divergent linéairement.

En renormalisant avec la constante λ_4, c'est-à-dire soustrayant le tenseur $k'_\mu M_{\mu\nu\varrho\sigma}(0000)$ on peut se ramener à manipuler des expressions seulement logarithmiquement divergentes et montrer que

$$k_1^\mu M^{\text{Ren.}}_{\mu\nu\varrho\sigma}(k_1k_2k_3k_4) = 0\,.$$

Ce résultat peut être aussi compris plus simplement de la façon suivante: la différence des deux expressions (7) est de la forme:

$$(A) = \int \frac{d^4p\cdot p^3}{((p+q)^2+a^2)^3} - \int \frac{d^4p(p+k)^3}{((p+q+k)^2+a^2)^3}\,,$$

après suppression des termes, au plus divergents logarithmiques, qui se compensent (A) peut aussi s'écrire $I[p+q] - I[p+q+k]$ avec $I[p] = \int d^4p\cdot p^3/(p^2+a^2)^3$. On calcule alors que: $I[p+q] = I[p]$ + terme linéaire en q; donc $k_1^\mu M_{\mu\nu\varrho\sigma}(k_1k_2k_3k_4)$ est un tenseur linéaire dans les impulsions k_1, k_2, k_3, k_4.

Supposons maintenant que nous ayons cherché le terme d'interaction entre trois potentiels quelconques φ'_ν, φ''_ϱ, φ'''_σ, et un potentiel de jauge $\varphi_\mu = \partial_\mu\Phi$; dans l'espace des impulsions nous aurions calculé

$$\Phi(k_1)\varphi_\nu(k_2)\varphi_\varrho(k_3)\varphi_\sigma(k_4)(k_1^\mu M^{(k_1k_2k_3k_4)}_{\mu\nu\varrho\sigma})\,,$$

c'est-à-dire, précisément, l'expression que nous venons de calculer. Or le seul covariant formé avec les quatre champs, et ne contenant qu'une seule dérivée,

est $(\partial_\mu \Phi)\varphi^\mu \varphi_\sigma \varphi^\sigma$. Cela démontre qu'une seule constante λ_4 suffit à cet ordre de la théorie de perturbation pour rendre le tenseur d'interaction de quatre photons invariant de jauge générale.

3. – Relations entre les divers termes de violation de l'invariance de jauge classique.

On peut poursuivre comme nous venons de le faire, l'étude du terme d'interaction entre photons et montrer, au moins au plus bas ordre de la théorie de perturbation, c'est-à-dire sans corrections radiatives, que les interactions entre plus de quatre photons sont invariantes de jauge. Cette étude générale peut être plus simplement faite dans l'espace des x. Soit $S_F^\Phi(x, x')$ le propagateur d'électron, sans interaction avec le champ de photon quantifié, mais en interaction avec un potentiel classique de jauge $\varphi_\mu = \partial_\mu \Phi$. D'après l'invariance théorique de jauge du propagateur d'électron, qui est manifeste tant qu'il n'y a pas de corrections radiatives, il est équivalent pour obtenir $S_\Phi(x, x')$ de calculer le propagateur de l'électron dans la jauge $\tilde{\psi} = \psi \exp[ie\Phi]$ on a donc

$$S^\Phi_{F\beta\alpha}(x', x) = 2\langle 0 \,|\, T\big(\exp[-ie\Phi(x)]\overline{\psi}_\alpha(x), \exp[ie\Phi(x')]\psi_\beta(x')\big) \,|\, 0\rangle$$
$$= \exp[ie(\Phi' - \Phi)] S_{F\beta\alpha}(x' - x)$$

et aussi

$$S^1_{\Phi\beta\alpha}(x', x) \equiv \langle 0 \,|\, [\overline{\psi}_\alpha(x), \psi_\beta(x')] \,|\, 0\rangle_\Phi = \exp[ie(\Phi' - \Phi)] S^1_{\beta\alpha}(x' - x) \,.$$

Les termes d'interaction entre un nombre quelconque de photons classiques de jauge nous sont donnés par le développement de l'expression

$$I[\Phi] = \partial_\mu \Phi(x) \langle 0 \,|\, j^\Phi_\mu(x) \,|\, 0\rangle \,.$$

Rigoureusement, cette expression n'a pas de sens mathématique défini parce que $S^1(0)$ est infini. Pour lui donner un sens nous généraliserons la méthode de limite de Dirac-Heisenberg [16] en définissant

$$\langle 0 \,|\, j^\Phi_\mu(x) \,|\, 0\rangle = \frac{ie}{8} \lim_{\xi \to 0} \langle 0 \,|\, \big([\overline{\psi}(x + \xi), \gamma_\mu \psi(x)] + [\overline{\psi}(x), \gamma_\mu \psi(x + \xi)] +$$
$$+ [\overline{\psi}(x - \xi), \gamma_\mu \psi(x)] + [\overline{\psi}(x), \gamma_\mu \psi(x - \xi)]\big) \,|\, 0\rangle =$$
$$= \frac{ie}{8} \lim_{\xi \to 0} \mathrm{Tr}\, \big[\gamma_\mu \big(\exp[-ie\big(\Phi(x + \xi) - \Phi(x)\big)] S_1(-\xi) +$$
$$+ \exp[ie\big(\Phi(x + \xi) - \Phi(x)\big)] S_1(\xi) + \exp[-ie\big(\Phi(x - \xi) - \Phi(x)\big)] S_1(\xi) +$$
$$+ \exp[ie\big(\Phi(x - \xi) - \Phi(x)\big)] S_1(-\xi)\big)\big] \,,$$

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[16] P. A. M. Dirac: *Proc. Camb. Phil.*, **30**, 150 (1934); W. Heisenberg: *Zeit. Phys.*, **90**, 209; **92**, 692 (1934); J. G. Valatin: *Proc. Roy. Soc.*, A **222**, 93 (1954).

on a

$$\eta_\mu(\xi) \equiv \mathrm{Tr}\,(\gamma_\mu S_1(\xi)) = 4\,\frac{\partial}{\partial\xi_\mu}\,\Delta_1(\xi) = -\,4\,\frac{\xi_\mu}{\pi^2}\,\frac{1}{\xi^2}\left(\frac{1}{\xi^2}-\frac{m^2}{4}\right) +$$

$$+ \text{termes reguliers si } \xi \to 0\,,$$

$$U(ie,\,\xi,\,\Phi) \equiv \exp\left[ie(\Phi(x+\xi)-\Phi(x))\right] =$$

$$= 1 + \xi_\mu(ie\,\partial_\mu\Phi) + \xi_\mu\xi_\nu\left(\frac{ie}{2}\,\partial_{\mu\nu}\Phi + \frac{(ie)^2}{2}\,\partial_\mu\Phi\partial_\nu\Phi\right) +$$

$$+ \xi_\mu\xi_\nu\xi_\varrho\left(\frac{ie}{3!}\,\partial_{\mu\nu\varrho}\Phi + \frac{(ie)^2}{2!}\,\partial_\mu\Phi\partial_{\nu\varrho}\Phi + \frac{(ie)^3}{3!}\,\partial_\mu\Phi\partial_\nu\Phi\partial_\varrho\Phi\right) + \text{termes en } \xi^4 + \ldots\,,$$

puisque $\eta_\mu(\xi) = -\eta_\mu(-\xi)$ on a:

$$\langle 0\,|\,j_\alpha^\Phi(x)\,|\,0\rangle = \frac{ie}{8}\,(\eta_\alpha(\xi)U(ie,\,\xi,\,\Phi) + \eta_\alpha(-\xi)U(-ie,\,\xi,\,\Phi) + \eta_\alpha(-\xi)U(ie,\,-\xi,\,\Phi) +$$

$$+ \eta_\alpha(\xi)U(-ie,\,-\xi,\,\Phi)) =$$

$$= \frac{ie}{8}\,\eta_\alpha(\xi)\left[U(ie,\,\xi,\,\Phi) - U(-ie,\,\xi,\,\Phi) + U(-ie,\,-\xi,\,\Phi) - U(ie,\,-\xi,\,\Phi)\right],$$

$$(8)\quad \langle 0\,|\,j_\alpha^\Phi(x)\,|\,0\rangle = \left(\frac{ie}{2}\right)\eta_\alpha(\xi)\left[ie\xi_\mu\partial_\mu\Phi + \xi_\mu\xi_\nu\xi_\varrho\left(\frac{ie}{3!}\,\partial_{\mu\nu\varrho}\Phi + \frac{(ie)^3}{3!}\,\partial_\mu\Phi\partial_\nu\partial_\varrho\Phi\right) - \ldots\right],$$

si nous faisons la moyenne sur les angles, comme on le fait plus couramment dans l'espace des impulsions, on a:

$$\xi_\mu\xi_\nu f(\xi^2) = \tfrac{1}{4}\,\delta_{\mu\nu}\xi^2 f(\xi^2)\,,$$

et

$$\xi_\mu\xi_\nu\xi_\varrho\xi_\sigma f(\xi^2) = \frac{1}{24}\,(\delta_{\mu\nu}\delta_{\varrho\sigma} + \delta_{\mu\sigma}\delta_{\nu\varrho} + \delta_{\mu\varrho}\delta_{\nu\sigma})(\xi^2)^2 f(\xi^2)\,,$$

on a finalement

$$I = \lim_{\xi\to 0}\left(-(ie)^2\,\frac{2}{\pi^2}\left(\frac{1}{(\xi^2)^2}-\frac{m^2}{4\xi^2}\right)\left[\partial_\varkappa\Phi\cdot\partial_\nu\Phi\cdot\frac{\xi^2\delta_{\nu\varkappa}}{4} + \frac{1}{3!}\cdot\frac{1}{1!}\,(\delta_{\mu\nu}\delta_{\varrho\sigma}+\delta_{\mu\sigma}\delta_{\nu\varrho}+\delta_{\mu\varrho}\delta_{\nu\sigma})(\xi^2)^2 \cdot\right.\right.$$

$$\left.\left.\cdot\{\partial_\sigma\Phi\,\partial_{\mu\nu\varrho}\Phi - e^2\partial_\sigma\Phi\cdot\partial_\mu\Phi\cdot\partial_\nu\Phi\cdot\partial_\varrho\Phi\}\right]\right) =$$

$$-\frac{2e^2}{\pi^2}\left[\frac{1}{4}\left(\frac{1}{(0)^2}-\frac{m^2}{4}\right)\partial_\mu\Phi\partial^\mu\Phi + \frac{1}{2\cdot 4!}\,\partial_\mu\Phi\partial^\mu\Box\Phi - \frac{e^2}{2\cdot 4!}\,(\partial_\mu\Phi\partial^\mu\Phi)^2\right],$$

puisque le potentiel $\partial_\mu\Phi$ n'est qu'un cas particulier d'un potentiel général q_μ, on doit donc avoir, à l'approximation considérée, les termes non invariants

pratique de jauge

$$I=\frac{2\alpha}{\pi}\left[\left(\frac{1}{0^2}-\frac{m^2}{4}\right)\varphi_\mu\varphi^\mu+\frac{1}{2\cdot 3!}\varphi_\mu\Box\varphi^\mu-\frac{\alpha\pi}{3}(\varphi_\mu\varphi^\mu)^2\right], \qquad \alpha=\frac{e^2}{4\pi}.$$

Ce résultat montre d'une façon simple l'existence de la constante d'interaction de quatre photons, mais met aussi en évidence sa relation avec la constante finie, B, non invariante pratique de jauge, qui accompagne la constante de masse du photon.

4. – L'invariance de jauge du lagrangien.

Nous pouvons maintenant construire une théorie qui possède les propriétés voulues d'invariance pratique de jauge classique restreinte, c'est-à-dire, telle que:

a) le photon a une masse nulle;

b) les tenseurs d'interaction entre photons s'annulent lorsque l'un des potentiels est un gradient de divergence nulle.

Ces propriétés sont en effet réalisées, si en plus du lagrangien $\mathscr{L}_1$, identique au lagrangien classique de Maxwell-Dirac, et des contre-termes de Dyson $\delta\mathscr{L}_1$, nous ajoutons les contre-termes de formes suivantes:

$$\delta\mathscr{L}_2=\lambda_0+\lambda_1^\mu\varphi_\mu+\lambda_2\varphi_\mu\varphi^\mu+\ldots\lambda_n(\varphi_\mu\varphi^\mu)^{n/2}+\ldots+\sum h_i(\partial_\mu\varphi^\mu)^i \tag{9}$$

qui sont déterminables en principe, à partir des hypothèses *a*) et *b*). Nous ne pouvons limiter, à priori, les constantes λ_i à l'ordre $(i=4)$. Pour l'invariance restreinte les constantes h_i peuvent être arbitraires.

Nous avons aussi soustrait, pour être complets, l'énergie du vide, λ_0. Sous de telles conditions les relations de Ward sont alors satisfaites, donc $Z_1=Z_2$, et $\delta\mathscr{L}_1$ possède alors la même invariance théorique que $\mathscr{L}_1$.

Puisque le lagrangien $\mathscr{L}_1+\delta\mathscr{L}_1+\delta\mathscr{L}_2$ est, par construction, tel que les propriétés d'invariance pratique de jauge en résultent, nous pouvons nous demander s'il peut être invariant sous le groupe $((1)+(2))$ et en particulier puisque $\mathscr{L}_1$ et $\delta\mathscr{L}_1$ le sont, si $\delta\mathscr{L}_2$ peut l'être aussi. Cela ne pourrait être le cas que si les constantes λ_i étaient nulles, ce qui est exclu, ou bien si les « constantes » λ_i n'étant pas invariantes de jauge sous le groupe $((1)+(2))$ satisfont la relation d'invariance:

$$\sum\lambda_i\cdot(\varphi)^i=\sum\tilde{\lambda}_i[\Phi]\cdot(\varphi+\partial\Phi)^i. \tag{10}$$

Cette équation signifie que, si l'on change la jauge des potentiels $\varphi_\mu \to$ $\to \varphi_\mu + \partial_\mu \Phi$, et simultanément la jauge des spineurs $\psi \to \psi \exp[ie\Phi]$ les « constantes » λ_i sont des fonctions qui dépendent de cette dernière jauge et varient avec elle, de telle sorte que l'identité précédente soit satisfaite.

Il convient de remarquer, que l'invariance de $\delta \mathscr{L}_2$ suppose l'invariance des constantes Z_1, Z_2, δm et Z_3 sous le groupe $((1)+(2))$; c'est-à-dire, explicitement $Z_1[\varphi_\mu, \psi] = Z_1[\varphi_\mu + \partial_\mu \Phi, \psi \exp[ie\Phi]]$, par conséquent $Z_1[\tilde{\varphi}_\mu - \partial_\mu \Phi, \psi] =$ $= Z_1[\tilde{\varphi}_\mu, \psi \exp[ie\Phi]]$: c'est une propriété qui se voit immédiatement en developpant les deux expressions en perturbation par rapport au potentiel $\partial_\mu \Phi$.

Cette invariance sous le groupe $((1)+(2))$ des constantes Z_1 et δm ne doit pas être confondue avec la non invariance éventuelle [17] de la constante Z_1 sous le seul groupe (2), classique ou quantique.

5. – Cohérence de l'hypothèse de la dépendance de jauge des constantes de renormalisation des interactions entre photons.

L'hypothèse de la dépendance de jauge des constantes λ_i permettant de résoudre le paradoxe de l'invariance de jauge, nous devons vérifier que cette circonstance se produit effectivement dans les calculs explicites, en perturbation ou globaux, de ces constantes. En théorie de perturbation, reportons nous au calcul du plus bas ordre de la polarisation du vide du photon (**2-*b***). Un changement de jauge constante des éléctrons, a pour effet de remplacer $S(p)$ par $S(p+eA)$, c'est-à-dire de produire un déplacement de la variable d'intégration. Ce changement est sans effet sur le tenseur invariant pratique de jauge qui ne diverge que logarithmiquement. Au contraire la partie non invariante pratique de jauge de ce tenseur, divergente quadratiquement, est affectée par ce déplacement de variable (*). Il est inutile de donner ici ce calcul puisqu'il est pratiquement contenu dans l'expression (8); en effet on a

$$\delta\mu_1^{(2)}(\Phi) \sim \mathrm{Tr}\,\gamma_\mu S_F^\Phi \gamma_\mu S_F^\Phi \sim \frac{\delta}{\delta \partial_\mu \Phi} \mathrm{Tr}\,(\gamma_\mu S_F^\Phi) \sim \frac{\delta}{\delta \partial_\mu \Phi} \langle 0 | j_\mu^\Phi(x) | 0 \rangle .$$

Nous allons maintenant calculer l'expression de la masse nue du photon suivant la méthode globale de Lehmann [18]. Le lagrangien que nous prendrons sera celui de Dyson, complété de trois termes: de masse du photon, d'interaction photon-photon, et d'un terme en h_2 déjà utilisé par Källén [19].

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[17] K. Johnson: *Phys. Rev.*, **112**, 1367 (1958); B. Zumino: *Nuovo Cimento*, **17**, 547 (1960).

(*) J. M. Jauch et F. Rohrlich: p. 457, ref. [6].

[18] G. Källén: *Helv. Phys. Acta*, **25**, 417 (1952); H. Lehmann: *Nuovo Cimento*, **11**, 342 (1954).

[19] G. Källén: p. 346, réf. [7].

Tant que nous n'exigeons que l'invariance de jauge constante, nous pouvons en effet choisir les constantes h_i à priori.

Si $h_{i>2} \neq 0$, les équations du mouvement seraient de degré $i > 2$, ce qui causerait certaines complications pour utiliser le schéma canonique. Nous faisons donc l'hypothèse que seul h_2 est différent de zero, et de plus different de Z_3, évitant ainsi la complication usuelle de l'élimination de $\partial_4\varphi_4$. Nous faisons de plus la seconde hypothèse que $\lambda_{i>4} = 0$. Ces deux hypothèses, dont la première est toujours loisible, tant que l'on n'exige que l'invariance de jauge constante, mais dont la seconde pourrait être fausse, seront en fait démontrée être correctes, même dans le cas de l'invariance de jauge classique générale.

Notre lagrangien s'écrit alors:

$$\mathscr{L} = -Z_2\overline{\psi}\left(\gamma\partial + m + \frac{\delta m}{Z_2}\right)\psi - \frac{1}{2}Z_3\partial_\mu\varphi_\nu\partial^\mu\varphi^\nu + \frac{1}{2}h(\partial_\mu\varphi^\mu)^2 +$$

$$+ ieZ_1\overline{\psi}\gamma_\mu\psi\varphi^\mu - ieZ_1\langle\overline{\psi}\gamma_\mu\psi\rangle_0\varphi^\mu - \frac{\delta\mu^2}{2}\varphi_\mu\varphi^\mu + \lambda_4(\varphi_\mu\varphi^\mu)^2 .$$

les équations du mouvement sont:

$$Z_3\Box\varphi_\mu - h\partial_\mu\partial_\nu\varphi^\nu = -ieZ_1\overline{\psi}\gamma_\mu\psi + ieZ_1\langle\overline{\psi}\gamma_\mu\psi\rangle_0 + \delta\mu^2\varphi_\mu - 4\lambda_4\varphi_\mu\varphi_\nu\varphi^\nu .$$

Nous utilisons les notations

$$x_4 = it \quad \varphi'_\nu = \varphi_\nu(x') , \qquad \dot{\varphi}_\nu = \frac{\partial}{\partial t}\varphi_\nu(x) , \qquad \overline{X} = (x_1x_2x_3) ,$$

$$\mu = 1, 2, 3, 4; \qquad i = 1, 2, 3 .$$

Des relations canoniques

$$[\varphi_\mu, \varphi'_\nu]_{t=t'} = 0 , \qquad [\psi_\alpha, \varphi'_\nu]_{t=t'} = 0 , \qquad [\psi_\alpha, \pi'_\nu]_{t=t'} = 0 ,$$

$$[\pi_\mu, \varphi'_\nu]_{t=t'} = -\delta_{\mu\nu}\delta^3(\overline{X} - \overline{X}') , \qquad [\pi_\mu, \pi'_\nu]_{t=t'} = 0 ,$$

avec

$$\pi_\mu = \frac{\partial\mathscr{L}}{\partial(\partial_4\varphi_\mu)} \qquad c\cdot a\cdot d = \pi_i = Z_3\partial_4\varphi_i , \qquad \pi_4 = (Z_3 - h)\partial_4\varphi_4 - h\partial_i\varphi^i ,$$

on déduit

$$[\partial_4\varphi_i, \varphi'_j]_{t=t'} = -\frac{\delta_{ij}}{Z_3}\delta^3(\overline{X} - \overline{X}') = [\partial'_4\varphi'_j, \varphi_i]_{t=t'} ,$$

$$[\partial_4\varphi_4, \varphi'_\mu]_{t=t'} = -\frac{\delta_{\mu 4}}{Z_3 - h}\delta^3(\overline{X} - \overline{X}') ,$$

c'est-à-dire:

$$[\partial_4\varphi_\mu,\ \varphi'_\nu]_{t=t'} = -\frac{\delta^3(\bar{X}-\bar{X}')}{Z_3}\left[\delta_{\mu\nu}+\frac{h\delta_{\mu 4}\delta_{\nu 4}}{Z_3-h}\right] = -\delta^3(\bar{X}-\bar{X}')\xi_{\mu\nu}\,,$$

$$[\partial_4\varphi_\mu,\ \partial_4\varphi'_\nu]_{t=t'} = \frac{h}{Z_3(Z_3-h)}(\delta_{\mu 4}\hat{c}_\nu - \delta_{\nu 4}\hat{c}_\mu)\delta^3(\bar{X}-\bar{X}')\,, \qquad (\hat{c}_1\delta^3(\bar{X})=0)\,.$$

Posant selon l'argument de Lehmann

$$\langle 0|[\varphi_\mu,\ \varphi'_\nu]|0\rangle = i\int(\varrho_1(a)\delta_{\mu\nu}+\varrho_2(a)\partial_\mu\partial_\nu)\Delta(x-x',\ a)\,\mathrm{d}a\,, \tag{11}$$

avec

$$(\Box - a)\,\Delta(x,\ a) = 0\,, \qquad \partial_4\Delta = i\,\delta^3(\bar{X})$$

on déduit des relations précédentes

$$\int\varrho_2\,\mathrm{d}a = 0\,, \qquad \int\varrho_1\,\mathrm{d}a = \frac{1}{Z_3}\,, \qquad \int(\varrho_1+a\varrho_2)\,\mathrm{d}a = \frac{1}{Z_3-h}\,, \tag{12}$$

c'est-à-dire aussi

$$\int\varrho_2 a\,\mathrm{d}a = \frac{h}{Z_3(Z_3-h)}\,.$$

Appliquant d'abord l'opérateur $Z_3\,\Box\delta_{\mu\nu} - h\hat{c}_\mu\hat{c}_\sigma - \delta\mu^2\delta_{\mu\sigma}$ sur la formule (11) on obtient, d'après l'équation du mouvement et les relations ci-dessus,

$$\langle 0|[-4\lambda\varphi_\mu\varphi^2 - ieZ_1\,\overline{\psi}\gamma_\mu\psi + ieZ_1\langle\overline{\psi}\gamma_\mu\psi\rangle_0,\ \varphi'_\nu]|0\rangle_{t=t'} = 0\,.$$

Dérivant ensuite par rapport à t', et prenant sa valeur pour $t=t'$, on obtient, après simplification par les relations précédentes,

$$4\lambda_4\langle 0|[\varphi_\sigma\varphi^\mu\varphi_\mu,\ \dot{\varphi}'_\nu]|0\rangle = -i\delta^3(\bar{X}-\bar{X})\left(\delta_{\nu\sigma}\int\varrho_1(a)[aZ_3-\delta\mu^2]\,\mathrm{d}a + \right.$$

$$\left. + \delta_{\nu 4}\delta_{\sigma 4}\int(\varrho_2(a)a[(Z_3-h)a-\delta\mu^2] - ha\varrho_1(a))\,\mathrm{d}a\right),$$

on a d'autre part:

$$\langle 0|[\varphi_\sigma\varphi_\mu\varphi^\mu,\ \dot{\varphi}'_\nu]|0\rangle = i\,\delta^3(\bar{X}-\bar{X}')(2\xi_{\mu\nu}\langle 0|\varphi_\sigma\varphi_\mu|0\rangle + \xi_{\nu\sigma}\langle 0|\varphi_\mu\varphi^\mu|0\rangle)\,,$$

ce qu'on peut calculer en prenant:

$$\langle 0|\varphi_\sigma\varphi_\mu|0\rangle = \lim_{x\to x'}\langle 0|\,T(\varphi_\sigma(x),\ \varphi_\mu(x'))\,|0\rangle$$

et on a:

$$\langle 0 | T(\varphi_\sigma, \varphi^\sigma) | 0 \rangle = \frac{-i}{(2\pi)^4} \iint \frac{4\varrho_1 + a\varrho_2}{p^2 + a - i\varepsilon} \, \mathrm{d}a \, \mathrm{d}^4p \,, \tag{13}$$

on obtient, lorsque les indices (σ et ν) égalent (i, i) ou (4, 4) respectivement, les relations:

$$\int \varrho_1 (Z_3 a - \delta\mu^2) \, \mathrm{d}a = -\frac{4\lambda_4}{Z_3} (2\langle 0 | \varphi_1 \varphi_1 | 0 \rangle + \langle 0 | \varphi_\mu \varphi^\mu | 0 \rangle) \,, \tag{14}$$

$$\int (\varrho_1 + a\varrho_2) [(Z_3 - h)a - \delta\mu^2] \, \mathrm{d}a = -\frac{4\lambda_4}{Z_3 - h} (2\langle 0 | \varphi_4 \varphi_4 | 0 \rangle + \langle 0 | \varphi_\mu \varphi^\mu | 0 \rangle) \,, \tag{15}$$

les combinant et tenant compte des relations (12) et (13), on a finalement

$$-4\delta\mu^2 + 4Z_3^2 \int \varrho_1 a \, \mathrm{d}a + (Z_3 - h)^2 \int a^2 \varrho_2 \mathrm{d}a - 2Z_3 h \int a\varrho_1 \, \mathrm{d}a + h^2 \int a\varrho_1 \, \mathrm{d}a =$$

$$= -6 \cdot 4 \cdot \lambda \langle 0 | \varphi_\mu \varphi^\mu | 0 \rangle = -4! \, \lambda_4 \left(\frac{-i}{(2\pi)^4} \right) \iint \frac{4\varrho_1 + a\varrho_2}{p^2 + a - i\varepsilon} \, \mathrm{d}a \, \mathrm{d}^4p \,,$$

ce que nous écrivons, en posant; $\mu_0^2 = \delta\mu^2/Z_3$, $\lambda_4^0 = \lambda_4/Z_3^2$

$$\mu_0^2 = Z_3 \int \varrho_1 a \, \mathrm{d}a - 4! \, \lambda_4^0 \frac{i}{(2\pi)^4} \iint \frac{Z_3 \varrho_1}{p^2 + a - i\varepsilon} \, \mathrm{d}a \, \mathrm{d}^4p + \tag{16}$$

$$+ \frac{1}{4} \left[\frac{(Z_3 - h)^2}{Z_3} \int a^2 \varrho_2 \, \mathrm{d}a - 2h \int a\varrho_1 \, \mathrm{d}a + \right.$$

$$\left. + \frac{h^2}{Z_3} \int \varrho_1 a \, \mathrm{d}a - 4! \, \lambda_4^0 Z_3 \frac{i}{(2\pi)^4} \iint \frac{\varrho_2 \cdot a}{p^2 + a - i\varepsilon} \, \mathrm{d}a \, \mathrm{d}^4p \right] .$$

On peut de la même façon évaluer les expressions

$$\langle 0 | [\partial_\mu j^\mu, \varphi_\nu'] | 0 \rangle_{t=t'} \quad \text{et} \quad \langle 0 | [\partial_\mu j^\mu, \dot{\varphi}_\nu'] | 0 \rangle_{t=t'}$$

que l'on trouve être nulles en vertu des relations précédentes, et en accord avec l'invariance de phase de la théorie.

Nous voyons donc, d'après (14), que si $\lambda_4 = 0$, $\mu_0^2 = \int \varrho_1^0 a \, \mathrm{d}a$ est invariant de jauge, comme on le croit généralement, puisque ϱ_1 l'est d'après sa définition.

Si, au contraire, γ_4 est différent de zéro, comme nous le trouvons en théorie de perturbation en attendant que l'on puisse le démontrer globalement, on a l'équation

$$\mu_0^2 = Z_3 \int \varrho_1 a \, \mathrm{d}a + \frac{4\lambda_4}{Z_3} (2\langle 0 | \varphi_1 \varphi_1 | 0 \rangle + \langle 0 | \varphi_\mu \varphi^\mu | 0 \rangle) \,.$$

Un changement de jauge $\varphi_\mu = \tilde{\varphi}_\mu + A_\mu$ change alors $\langle 0|\varphi_\mu\varphi^\mu|0\rangle$ en $\langle 0|\tilde{\varphi}_\mu\tilde{\varphi}_\mu|0\rangle + 2A_\mu\langle 0|\tilde{\varphi}_\mu|0\rangle + A_\mu A^\mu$; le second membre de l'équation précédente n'est donc pas invariant et par conséquent μ_0^2 non plus.

L'invariance de jauge de l'équation du photon résulte, dans le premier cas de la compensation entre la variation du courant du vide et la variation du potentiel intervenant dans le terme de masse lors d'un changement de jauge $(I+2)$, alors que dans le second cas, la variation de la masse compense par ailleurs les termes de variation du courant du vide en A^3 ainsi que les termes en λ_4 provenant de la variation du potentiel.

6. – Calcul des constantes λ_i.

La relation d'invariance de jauge (10) pour $\partial_\mu \Phi = A_\mu = \text{constantes}$, donne

$$\sum \lambda_i \cdot \varphi^i = \sum \tilde{\lambda}_i(A) \cdot (\varphi + A)^i$$

qui peut s'écrire aussi

$$\sum \lambda_i (\tilde{\varphi} - A)^i = \sum \tilde{\lambda}_i(A) \cdot (\tilde{\varphi})^i \,. \tag{17}$$

Introduisons les tenseurs symétriques $\gamma_i^{\mu\nu\ldots}$ et $\tilde{\gamma}_i^{\mu\nu\ldots}$ tels que

$$\sum \tilde{\lambda}_i \tilde{\varphi}^i \equiv \tilde{\gamma}_0 + \tilde{\gamma}_1^\mu \tilde{\varphi}_\mu + \frac{1}{2!}\tilde{\gamma}_2^{\mu\nu}\tilde{\varphi}_\mu\tilde{\varphi}_\nu + \ldots + \frac{1}{n!}\tilde{\gamma}_n^{\mu\nu\ldots}(\tilde{\varphi}_\mu\tilde{\varphi}_\nu \ldots \tilde{\varphi}_{\alpha_n}) + \ldots ,$$

$$\sum \lambda_i (\tilde{\varphi} - A)^i = \gamma_0 + \gamma_1^\mu (\tilde{\varphi} - A)_\mu + \ldots ,$$

donc

$$\frac{1}{2!}\gamma_2^{\mu\nu} = \lambda_2 \delta_{\mu\nu}; \quad \frac{1}{4!}\gamma_4^{\mu\nu\varrho\sigma} = \frac{1}{3}\lambda_4(\delta_{\mu\nu}\delta_{\varrho\sigma} + \delta_{\mu\sigma}\delta_{\nu\varrho} + \delta_{\mu\varrho}\delta_{\nu\sigma});$$

$$\frac{1}{6!}\gamma_6^{\lambda\mu\nu\varrho\sigma} = \frac{1}{15}\lambda_6(\delta_{\lambda\mu}\delta_{\nu\varrho}\delta_{\sigma\tau} + \ldots);$$

Supposant A_μ infinitésimal et identifiant les termes en $\tilde{\varphi}_\mu\tilde{\varphi}_\nu \ldots$ dans les deux membres de (17) on obtient les relations

$$\gamma_p^{\mu\nu\ldots} = (-1)^p \left(\frac{\partial^p}{\partial A_\mu \partial A_{\nu\ldots}} \tilde{\gamma}_0(A)\right)_{A=0}; \quad \gamma_{p+1}^{\mu\nu\varrho\ldots} = (-1)^p \left(\frac{\partial^p}{\partial A_\nu \partial A_{\sigma\ldots}} \tilde{\gamma}_1^\mu(A)\right)_{A=0}; \ldots ,$$

que nous pouvons écrire symboliquement

$$\gamma_{p+n} = (-1)^p \left(\partial^p \tilde{\gamma}_n(A)\right)_0 . \tag{17'}$$

Par conséquent, les constantes λ_i peuvent se calculer à partir des expressions de la densité d'énergie du vide, ou de la densité de courant du vide dans un potentiel externe.

On a en particulier

$$\lambda_2 \delta_{\mu\nu} = \frac{1}{2!}\gamma_2^{\mu\nu} = \frac{1}{2!}(\partial^{\mu\nu}\tilde{\gamma}_0)_0 = -\frac{1}{2!}(\partial^\nu \tilde{\gamma}_1^\mu)_0 ,$$

$$\lambda_4(\delta_{\mu\nu}\delta_{\varrho\sigma} + \delta_{\mu\sigma}\delta_{\nu\varrho} + \delta_{\mu\varrho}\delta_{\nu\sigma}) = \frac{3}{4!}\gamma_4^{\mu\nu\varrho\sigma} = \frac{3}{4!}(\partial^{\mu\nu\varrho\sigma}\tilde{\gamma}_0)_0 = -\frac{3}{4!}(\partial^{\nu\varrho\sigma}\tilde{\gamma}_1^\mu)_0 .$$

Le calcul de λ_0 qui est la qualité la plus fondamentale de la théorie et la déduction des constantes λ_i à partir de $\tilde{\lambda}_0(A)$ ne sera pas donné ici à cause de la longueur de cet exposé (*). Nous déduirons ici les constantes $\lambda_{i>1}$ de la fonction $\lambda_1^\mu(A)$ que nous allons maintenant déterminer directement.

En l'absence d'un potentiel gradient externe, l'équation du mouvement du champ de photon (non renormalisé) φ_μ^0 est, avec ${}_0\gamma_n \equiv \gamma_n \cdot (Z_3)^{-n/2}$ et $h_0 = h/\sqrt{Z_3}$.

$$\Box\varphi_0^\mu - h_0 \partial^\mu \partial_\nu \varphi_\nu^0 = J_0^\mu \equiv ie_0 \overline{\psi}_0 \gamma^\mu \psi_0 - {}_0\gamma_1^\mu - {}_0\gamma_2^{\mu\nu}\varphi_\nu^0 - \ldots - \frac{1}{(n-1)!}\,{}_0\gamma_n^{\mu\nu\ldots}\overbrace{(\varphi_\nu^0\varphi_\varrho^0\ldots)}^{(n-1)} ,$$

ce qui définit un opérateur de courant (non renormalisé), J_0^μ. Ce courant satisfait l'équation de conservation $\partial_\mu J_0^\mu = 0$.

On obtient aisément la forme de J_0^μ avec la méthode utilisée par GELL-MANN et LÉVY (20) en calculant l'expression de $\partial\mathscr{L}/\partial A_\mu^0$ produite par la variation $\psi \to \psi(1 + ieA_\mu^0 x^\mu)$ qui entraine

$${}_0\gamma_{\sigma\tau\ldots}^{(n)} \to {}_0\gamma_{\sigma\tau\ldots}^{(n)} + \frac{\partial\, {}_0\gamma_{\sigma\tau}^{(n)}}{\partial A_\mu^0} A_\mu^0 = {}_0\gamma_{\sigma\tau\ldots}^{(n)} - {}_0\gamma_{\mu\sigma\tau\ldots}^{(n+1)} A^{0\mu} .$$

La constante ${}_0\gamma_1^\mu$, dont le rôle est de soustraire le courant du vide doit être telle que $\langle 0 | J_0^\mu | 0 \rangle = 0$:

$${}_0\gamma_1^\mu = ie_0 \langle 0 | \overline{\psi}_0 \gamma^\mu \psi_0 | 0 \rangle - \sum_{n=2}^{\infty} \frac{1}{(n-1)!}\,{}_0\gamma_n^{\mu\nu\varrho\ldots} \langle 0 | \overbrace{\varphi_\nu^0 \varphi_\varrho^0 \ldots}^{(n-1)} | 0 \rangle ;$$

(*) Un bref résumé de cette méthode est donné dans notre note réf. (*), p. 1.

(20) M. GELL-MANN et M. LÉVY: *Nuovo Cimento*, **16**, 705 (1960).

de cette série, seuls contribuent les termes contenant un nombre pair d'opérateurs de photon puisque

$$\langle 0 | \varphi_\nu^0 | 0 \rangle \sim \langle 0 | J_0^\mu | 0 \rangle = 0 ,$$

donc

$${}_0\gamma_1^\mu = ie_0 \langle 0 | \overline{\psi}_0 \gamma^\mu \psi_0 | 0 \rangle - \frac{1}{2} {}_0\gamma_3^{\mu\nu\varrho} \langle 0 | \varphi_\nu^0 \varphi_\sigma^0 | 0 \rangle - \frac{1}{4!} {}_0\gamma_5^{\mu\nu\varrho\sigma\tau} \langle 0 | \varphi_\nu^0 \varphi_\varrho^0 \varphi_\sigma^0 \varphi_\tau^0 | 0 \rangle - \dots .$$

D'autre part les constantes ${}_0\gamma_{2p+1}$ d'interaction de $2p+1$ photons sont nulles d'après la symétrie de charge. La constante ${}_0\gamma_1^\mu$ est donc égale à

$$ie_0 \langle 0 | \overline{\psi}_0 \gamma^\mu \psi_0 | 0 \rangle$$

et on peut la rendre nulle par le procédé de Heisenberg-Dirac (comme à la Section **3**).

D'après le principe d'invariance de jauge, et le fait que $Z_1 = Z_2$ le lagrangien du système des électrons et des photons placés dans un potentiel constant s'obtient en remplaçant φ_μ^0 par $\varphi_\mu^0 + A_\mu^0$, ou bien en changeant la jauge des spineurs, et aussi les constantes ${}_0\lambda_i$ en ${}_0\tilde{\lambda}_i(A^0)$.

C'est ce dernier point de vue que nous choisirons ici.

On a alors pour équation du mouvement du photon plongé dans le potentiel A_μ^0:

$$\Box \varphi_0^\mu - h_0 \partial^\mu \partial_\nu \varphi_0^\nu = \tilde{J}_0^\mu = ie_0 \tilde{\overline{\psi}}_0 \gamma^\mu \tilde{\psi} - \sum_1^\infty \frac{1}{(n-1)!} {}_0\gamma_n^{\mu\nu\varrho\dots} \overbrace{(\varphi_\nu^0 \varphi_\varrho^0 \dots)}^{(n-1)} ,$$

avec $\tilde{\psi}_0 = \psi_0 \exp[ie_0(A^0 x)]$.

Le courant dans le potentiel externe est $\tilde{J}_0^\mu$; la fonction ${}_0\tilde{\gamma}_1^\mu$ définie par la condition $\langle 0 | \tilde{J}_0^\mu | 0 \rangle = 0$ est alors:

$$(17'') \qquad {}_0\tilde{\gamma}_1^\mu(A) = ie_0 \langle 0 | \tilde{\overline{\psi}}_0 \gamma^\mu \tilde{\psi}_0 | 0 \rangle - \sum_2^\infty \frac{1}{(n-1)!} {}_0\gamma_n^{\mu\nu\dots} \langle 0 | \overbrace{\varphi_\nu^0 \dots}^{(n-1)} | 0 \rangle .$$

Voyons d'abord que seuls les termes impairs ${}_0\tilde{\gamma}_{2p+1}$ contribuent.

En effet, compte tenu de ce que $\langle 0 | q_0^\mu | 0 \rangle \sim \langle 0 | \tilde{J}_0^\nu | 0 \rangle = 0$, les éléments de matrice $\langle 0 | \overbrace{q_\mu^0 \dots}^{(2p+1)} | 0 \rangle$ sont indépendants du potentiel de jauge (constant) d'après le théorème de la Section **1**°B-2, donc nuls comme précédemment.

A la différence du cas où il n'y a pas de potentiel externe, les constantes ${}_0\tilde{\gamma}_{2p+1}$ ne sont généralement pas nulles. On a alors d'après (17')

$$- {}_0\gamma_2^{\mu\nu} = + (\partial_0^\nu \gamma_1^\mu)_0 = ie_0 \langle 0 | \frac{\partial}{\partial A_\nu^0} \tilde{\overline{\psi}}_0 \gamma_\mu \tilde{\psi}_0 | 0 \rangle - \sum_{q=1} \frac{1}{(2q)!} \left(\frac{\partial}{\partial A_\nu^0} {}_0\tilde{\gamma}_{2q+1}^{\mu\nu\dots} \right) \langle 0 | \overbrace{\varphi_\sigma^0 \varphi_\tau^0 \dots}^{(2q)} | 0 \rangle .$$

Les constantes λ_i satisfaisant par hypothèse les relations d'invariance de jauge constante (10) et le lagrangien étant invariant sous le groupe $((1)+(2))$, nous pouvons appliquer le théorème de Ward (restreint) pour calculer les expressions $(\partial_{\mu\nu} \dots {}_0\tilde{\gamma}_1)_0$.

Définissons

$$I_0^{\prime\prime}(A_0) \equiv ie_0 \langle 0 | \bar{\tilde{\psi}} \gamma'' \tilde{\psi}_0 | 0 \rangle = ie_0 \left(\exp [ie_0(A_0(x-x'))] \langle 0 | \bar{\psi}_0 \gamma_\mu \psi_0' | 0 \rangle \right)_{x \to x'}$$

où le champ ψ_0 satisfait aux équations du mouvement en l'absence de potentiel externe: $(\gamma\partial + m_0)\psi_0 = ie_0 \varphi_\mu^0 \gamma^\mu \psi_0$.

Nous pouvons alors exprimer $\langle 0 | \bar{\psi}_0 \gamma'' \psi_0 | 0 \rangle$ à l'aide de la représentation spectrale du type de Lehmann.

Nous définissons des fonctions $\sigma_1^0(a)$ et $\sigma_2^0(a)$ par:

$$\langle 0 | \{ \psi_0, \bar{\psi}_0' \} | 0 \rangle_{t=t'} = - i \int_0^{+\infty} (\sigma_1^0(a) \gamma\partial + \sigma_2^0(a) (\Delta(x-x', a) \, \mathrm{d}a \, ,$$

d'où

$$\langle 0 | \{ \psi_0, \bar{\psi}_0' \}_{t=t'} | 0 \rangle = \gamma_4 \delta^3(\bar{X} - \bar{X}') \int \sigma_1^0(a) \, \mathrm{d}a \, ,$$

d'où canoniquement

$$\int \sigma_1^0(a) \, \mathrm{d}a = 1 \, .$$

Soit

$$f_0 \equiv (\gamma\partial + m_0)\psi_0 = ie_0 \varphi_\mu^0 \gamma^\mu \psi_0 \, ,$$

on a:

$$\langle 0 | \{ f_0, \bar{\psi}_0' \} | 0 \rangle_{t=t'} \sim \langle 0 | \{ \varphi_\mu^0 \psi_0, \bar{\psi}_0' \} | 0 \rangle_{t=t'} \sim \langle 0 | \varphi_\mu^0 | 0 \rangle \sim \langle 0 | \tilde{J}_\mu^0 | 0 \rangle = 0 \, ,$$

puisque nous avons précisément soustrait le courant du vide dans notre lagrangien. On déduit alors, comme LEHMANN

$$m_0 = - \frac{\int \sigma_2^0 \, \mathrm{d}a}{\int \sigma_1^0 \, \mathrm{d}a} = - \int \sigma_2^0 \, \mathrm{d}a \, .$$

Calculons maintenant:

$$\langle 0 | \left\{ \varphi_\mu^0 \psi_0, \frac{\partial}{\partial x_4'} \bar{\psi}_0' \right\} | 0 \rangle_{t=t'} \, ,$$

en utilisant les équations du mouvement

$$\partial_4' \bar{\psi}_0' = - \partial_i \bar{\psi}_0' \gamma_i \gamma_4 + m_0 \bar{\psi}_0' \gamma_4 - ie_0 \varphi_0'^\nu \bar{\psi}_0' \gamma_\nu \gamma_4 \, .$$

Les termes en

$$\langle 0|\{\varphi^0_\mu\psi_0,\ (\partial_i\overline{\psi}'_0)\}|0\rangle_{t=t'} \quad \text{et} \quad \langle 0|\{\varphi^0_\mu\psi_0,\ \overline{\psi}'_0\}|0\rangle_{t=t'}$$

sont nuls car proportionnels à $\langle 0|q^0_\mu|0\rangle \sim \langle 0|\tilde{J}^0_\mu|0\rangle = 0$. Le seul terme à contribuer est de la forme:

$$\langle 0|\{\varphi^0_\mu\psi_{0\alpha},\ \varphi'^0_\nu\overline{\psi}'_{0\beta}\}|0\rangle_{t=t'} = \langle 0|\varphi^0_\mu\varphi'_{0\nu}\{\psi_{0\alpha},\ \bar{\psi}'_{0\beta}\}|0\rangle_{t=t'} = \langle 0|\varphi^0_\mu\varphi'^0_\nu|0\rangle_{t=t'}\gamma^4_{\alpha\beta}\delta^3(\overline{X}-\overline{X}'),$$

comme on le voit en tenant compte des relations canoniques de commutation des champs q^0_μ entre eux et avec les ψ_0 et $\overline{\psi}_0$, et des relations d'anticommutation des ψ_0 et $\overline{\psi}_0$.

On a donc

$$\langle 0|\left\{(\gamma\partial+m_0)\psi_\alpha,\ \frac{\partial}{\partial x'_4}\overline{\psi}'_{0\beta}\right\}|0\rangle_{t=t'} =$$

$$= i\left(\frac{\partial}{\partial x_4}\int\left((a\sigma^0_1+m_0\sigma^0_2)\delta_{\alpha\beta}+(m_0\sigma^0_1+\sigma^0_2)\gamma_{\alpha\beta}\partial\right)\Delta(x-x',a)\,\mathrm{d}a\right)_{t=t'} =$$

$$= -(\delta^3(\overline{X}-\overline{X}')\delta_{\alpha\beta}\int(a\sigma^0_1+m_0\sigma^0_2)\,\mathrm{d}a\,,$$

le premier membre donne alors

$$-(ie_0)^2(\gamma_\mu\gamma_4\gamma_\nu\gamma_4)_{\alpha\beta}\langle 0|\varphi^0_\mu\varphi'^0_\nu|0\rangle_{t=t'}\delta^3(\overline{X}-\overline{X}')$$

soit

$$-(e_0)^2\tfrac{1}{2}(1)_{\alpha\beta}\langle 0|\varphi^0_\mu\varphi'^{0\mu}|0\rangle_{x\to x'}\delta^3(\overline{X}-\overline{X}')$$

en tenant compte de

$$\langle 0|\varphi^0_\mu\varphi'^0_\nu|0\rangle_{t=t'} = \tfrac{1}{4}\langle 0|\varphi^0_\alpha\varphi^{0\alpha'}|0\rangle_{t=t'}\delta_{\mu\nu}\,.$$

On a donc

$$\int(a\sigma^0_1+m_0\sigma^0_2)\,\mathrm{d}a = \frac{1}{2}\,e_0^2\langle 0|\varphi^0_\mu\varphi^{0\mu}|0\rangle\,;$$

avec

$$m_0 = -\int\sigma^0_2\,\mathrm{d}a\,,$$

on obtient:

$$\int a\sigma^0_1\,\mathrm{d}a = m_0^2+\frac{1}{2}\,e_0^2\langle 0|\varphi^0_\mu\varphi^{0\mu}|0\rangle\,. \tag{18}$$

Nous pouvons maintenant calculer les expressions

$$\left(\frac{\partial^n}{\partial A^0_\nu \partial A^0_{\varrho\ldots}} \cdot I^\mu_0(A^0)\right)_{A^0=0} :$$

on a

$$\left(\frac{\partial}{\partial A^0_\nu} I^\mu_0(A_0)\right)_0 = (ie_0)^2 \big((x-x')_\nu \langle 0 \,|\, \mathrm{Tr}\,(\bar\psi_0 \gamma^\mu \psi'_0) \,|\, 0\rangle\big)_{x\to x'} =$$

$$= (ie_0)^2 \left((x-x')_\nu \frac{1}{2}\int \mathrm{d}a\ \mathrm{Tr}\,(\gamma_\mu\gamma_\alpha)\partial_\alpha \Delta_1(x-x', a)\sigma^0_1(a)\right)_{x\to x'};$$

c'est-à-dire en posant $\xi^2 = (x-x')^2$ et $(\xi_\mu\xi_\nu/\xi^2)_{\xi\to 0} = \frac{1}{4}\delta_{\mu\nu}$ et tenant compte de (18)

$$-\frac{(ie_0)^2}{2\pi^2}\delta_{\mu\nu}\int \sigma^0_1(a)\left[-\frac{1}{\xi^2}+\frac{a}{4}+O(\xi)\right]_{\xi\to 0}\mathrm{d}a =$$

$$= \frac{2\alpha_0}{\pi}\delta_{\mu\nu}\left(\infty^2 - \frac{m_0^2}{4}\right) - \alpha_0^2\langle 0\,|\,\varphi^0_\mu\varphi^{0\mu}\,|\,0\rangle\,\delta_{\mu\nu};$$

on a aussi de la même façon:

$$\left(\frac{\partial^3}{\partial A^0_\nu \partial A^0_\varrho \partial A^0_\sigma} I^\mu_0(A_0)\right)_0 = (ie_0)^4 \xi_\nu\xi_\varrho\xi_\sigma \cdot \frac{2}{\pi^2}\int \sigma^0_1(a)\,\frac{\xi_\nu}{\xi^2}\left(-\frac{1}{\xi^2}+\frac{a}{4}+O(\xi)\right)\mathrm{d}a =$$

$$= \alpha_0^2 \cdot \frac{4}{3}\,(\delta_{\mu\nu}\delta_{\varrho\sigma}+\delta_{\mu\sigma}\delta_{\nu\varrho}+\delta_{\mu\varrho}\delta_{\nu\sigma})\int \sigma^0_1(a)\left[-1+\frac{a\xi^4}{4}+\ldots\right]_{\xi\to 0}\mathrm{d}a =$$

$$= -\alpha_0^2\frac{4}{3}\,(\delta_{\mu\nu}\delta_{\varrho\sigma}+\delta_{\mu\sigma}\delta_{\nu\varrho}+\delta_{\mu\varrho}\delta_{\nu\sigma}),$$

et avec

$$\xi_\mu\xi_\nu\xi_\varrho\xi_\sigma\xi_\tau\xi_\lambda = \big((\xi^2)^3/3\times 4^3\big)\underbrace{(\delta_{\lambda\mu}\delta_{\nu\varrho}\delta_{\sigma\tau}+\ldots)}_{15\ \text{termes}},$$

$$\big(\partial^5_{\lambda\nu\varrho\sigma\tau} I^\mu_0(A^0)\big)_0 = -\alpha_0^3\cdot\frac{2}{3}\,\pi(\delta_{\lambda\mu}\delta_{\nu\varrho}\delta_{\sigma\tau}+\ldots)\left[\int \sigma_1(a)\left(-\xi^2+(\xi^2)^2\frac{a}{4}+\ldots\right)_{\xi\to 0}\mathrm{d}a\right] = 0,$$

$$\big(\partial^{2p+1} I^\mu_0(A^0)\big)_{0p\geq 4} = 0\,.$$

On a donc finalement, utilisant (17′) et (17″), les équations des ${}_0\gamma_{\ldots}$:

$$-\,{}_0\gamma'^{\mu\nu}_2 = -\,2!\,\lambda^0_2\delta_{\mu\nu} = \frac{2\alpha_0}{\pi}\,\delta_{\mu\nu}\left(\infty^2-\frac{m_0^2}{4}\right) - \alpha_0^2\langle 0\,|\,\varphi^\mu_0\varphi^{0\mu}\,|\,0\rangle -$$

$$+\frac{1}{2!}\,{}_0\gamma^{\mu\nu\varrho\sigma}_4\langle 0\,|\,\varphi^0_\varrho\varphi^0_\sigma\,|\,0\rangle + \frac{1}{4!}\,{}_0\gamma^{\mu\nu\varrho\sigma\alpha\beta}_6\langle 0\,|\,\varphi^0_\alpha\varphi^0_\beta\varphi^0_\varrho\varphi^0_\sigma\,|\,0\rangle\ldots,$$

$$-{}_0\gamma_4^{\mu\nu\varrho\sigma} = -\frac{4!}{3}\lambda_4^0(\delta_{\nu\mu}\delta_{\varrho\sigma}+\ldots) = +\alpha_0^2\frac{4}{3}(\delta_{\mu\nu}\delta_{\varrho\sigma}+\ldots)+\frac{1}{2!}{}_0\gamma_6^{\mu\nu\varrho\sigma\alpha\beta}\langle 0|\varphi_\alpha^0\varphi_\beta^0|0\rangle+\ldots,$$

$$-{}_0\gamma_6^{\mu\nu\varrho\sigma\tau\lambda} = -\frac{6!}{5\cdot 3}\lambda_6^0(\delta_{\mu\nu}\delta_{\varrho\sigma}\delta_{\lambda\tau}+\ldots) = O+\frac{1}{2!}{}_0\gamma_8^{\mu\nu\ldots}\langle 0|\varphi_\alpha^0\varphi_\beta^0|0\rangle+\ldots$$

. . . . . .

$$-{}_0\gamma_{2\nu}^{\mu\nu\ldots}{}_6 = O+{}_0\gamma_{2\nu+2}^{\mu\nu\ldots}\langle 0|\varphi_\alpha^0\varphi_\beta^0|0\rangle+\ldots.$$

Ces equations sont satisfaites par ${}_0\gamma_{2\nu\geqslant 6}$ (donc $\lambda_{2\nu\geqslant 6}^0$) $= 0$; alors

$$\lambda_4^0 = \alpha_0^2\frac{1}{3!}$$

et les termes en $\langle 0|\varphi_\mu^0\varphi^{0\prime\prime}|0\rangle$ se compensent dans l'equation de λ_2^0, qui donne finalement

$$-2\lambda_2^0 = \mu_0^2 = \frac{2\alpha_0}{\pi}\left(\infty^2-\frac{m_0^2}{4}\right) = \frac{\alpha_0}{2\pi}(\infty'^2-m_0^2).$$

Par conséquent *les constantes* λ_0^i $(i>1)$ *sont égales à leur valeur calculée au plus bas ordre de la théorie de perturbation en* α_0 *et* m_0.

Cependant nous avons fait au cours de la précédente démonstration certaine hypothèse implicite, sur laquelle il nous faut maintenant revenir.

Nous avons en effet obtenu pour la constante $(\hat{c}_{\nu\varrho\sigma}^3 I^\mu)_0$ l'expression

$$\sim\alpha_0^2\int\sigma_1^0(a)\left|\xi^2\left(-\frac{1}{\xi^2}+\frac{a}{4}+O(\xi^2)\right)\right|_{\xi^2\to 0}\mathrm{d}a\to\alpha_0^2\int\sigma_0^2\,\mathrm{d}a=\alpha_0^2.$$

Mais si nous avions fait l'intégration sur (a) avant le passage à la limite $(\xi\to 0)$ nous aurions obtenu, en plus du résultat précédent, des termes indéfinis $(\xi^2)_{\xi\to 0}\cdot\int\sigma_1^0(a)\cdot a\,\mathrm{d}a+\ldots$ si $\sigma_1^0(a)$ ne converge pas assez rapidement.

Cette ambiguité est levée si nous laissons ξ fini dans l'expression alors bien définie

$$I_0^\mu(A^0,\xi) = \frac{ie_0}{2}\int\sigma_1^0(a)\,\mathrm{Tr}\,(\gamma_\mu S_1^A(\xi,a))\,\mathrm{d}a.$$

Les expressions $(\hat{c}_\mu I_0^\mu)_0$, $(\hat{c}_{\mu\nu\varrho}I_0^\mu)_0$ sont alors respectivement proportionnelles à

$$\int\mathrm{d}a\int\exp[ip\xi]\,\mathrm{d}^4p\,\sigma_1^0(a)\left(\frac{2}{p^2+a-i\varepsilon}-\frac{p^2}{(p^2+a-i\varepsilon)^2}\right)$$

et

$$\int\mathrm{d}a\int\exp[ip\xi]\,\mathrm{d}^4p\,\sigma_1^0(a)\left(\frac{4a}{(p^2+a-i\varepsilon)^3}-\frac{3a^2}{(p^2+a-i\varepsilon)^4}\right),$$

Alors que la seconde intégrale ne nécessite que la convergence de $\int \sigma_1(a)\,da$ pour n'être pas ambigue suivant que l'on prenne la limite $\xi \to 0$ avant ou après l'intégration sur (a), la première intégrale, au contraire, est ambigue si $\int \sigma_1(a) a\,da$ ne converge pas, et sa valeur peut être différente suivant que l'on intègre sur (a) avant ou après que $\xi \to 0$.

Cette ambiguité apparait effectivement si le deuxième membre de la relation (18), $\langle 0 | q''_\mu q''_0 | 0 \rangle \sim \int da \int d^4p\, ((4\varrho_1^0 + \varrho_2^0)/(p^2 + a - i\varepsilon))$ diverge, ce qui est le cas si $\int \varrho_1 da$ et $\int \varrho_1 a\,da$ ne sont pas nuls, comme cela se produit en général. Le cas où $\varrho_2 = -4\varrho_1$, pour lequel le second membre serait nul, peut être réalisé par un changement de jauge quantique; mais alors la nouvelle fonction σ_1' (non invariante de jauge quantique) se déduit de σ_1 par une relation telle que la dépendance en ϱ_2 disparue du second membre de (18) réapparait au premier, comme cela doit être d'ailleurs puisque m_0^2 est invariant.

Un traitement qui lève l'ambiguité mentionnée plus haut conduit à des développements qui dépassent le cadre de cet article dont l'objet est principalement de résoudre le paradoxe de la masse du photon et de montrer l'importance des constantes λ_i. Nous nous limiterons donc ici à utiliser la prescription ($\xi \to 0$ avant l'interaction sur (a)); les conclusions qui en dépendent seront précisées et révisées éventuellement dans un article ultérieur.

Mentionnons cependant que les valeurs des constantes λ_i obtenues de cette façon s'interprètent très bien dans la théorie des graphes.

D'après les propriétés connues, d'une part de la renormalisation des graphes (en théorie de perturbation en α_0 et m_0), et d'autre part de l'invariance pratique de jauge, nous savons que, à toute simple boucle d'électron, B^{2N}, portant $2N$ nœuds et d'où partent $2N$ photons d'impulsion nulle, est associée dans la matrice S un contre-terme — la constante $\lambda_0^{2N} \sim \alpha_0^N$ (infinie, finie, ou conditionnellement nulle) —, qui renormalise cette boucle en la rendant identiquement nulle. Les diverses corrections radiatives à cette boucle sont constituées de boucles B^{2N+2K} d'où partent $2K$ lignes fermées de photons, éventuellement enchainées entre elles par d'autres boucles. Ces corrections sont renormalisées dans la matrice S par les contre-termes en $(\alpha_0^{N+K}, \alpha_0^{N+K+1}, \ldots \alpha_0^\infty)$ de λ_0^{2N}. Mais à chaque boucle B^{2N+2K} est aussi associé un contre-terme de λ_0^{2N-2K} en α_0^{N+K} dont le seul effet est précisément d'annuler la contribution éventuelle apportée par la boucle B^{2N-2K}. Ainsi les termes de λ_0^{2N} en α_0^{N+K} $(K > 0)$ sont nuls, et le seul terme, en α_0^N, est donné par le calcul de B^{2N}.

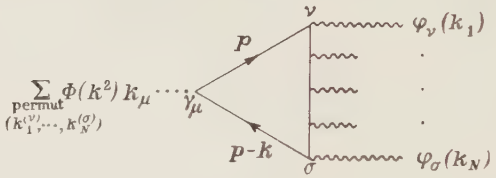
7. – Théorie invariante de jauge classique générale.

Nous nous sommes limités, dans les derniers chapitres à construire une théorie qui ne satisfait qu'à l'invariance de jauge constante. Nous allons maintenant montrer que, moyennant une modification mineure, cette même théorie

possède aussi l'invariance de jauge classique générale. La théorie précédente est construite de telle sorte que le théorème de Ward restreint soit valide, c'est-à-dire que les interactions des photons avec un photon d'impulsion nulle soient nulles. Nous voudrions nous assurer d'une théorie qui possède la propriété plus générale (Sect. 1° B-2), qui, pour être satisfaite, pourrait éventuellement nécessiter l'introduction de termes de couplage en $(\hat{c}_\mu q_\mu)^n$ ou $(\hat{c}_\mu \varphi^\mu)\,\Box^p(\hat{c}_\sigma \varphi^\sigma)$, que nous n'avons pas considérés jusqu'à présent.

Nous allons démontrer que les interactions de photons quelconques $q^\mu(k_i)$ avec un potentiel de jauge quelconque $\hat{c}_\mu \Phi(k)$, sont nulles lorsqu'on utilise les seuls contre-termes déjà introduits dans $\delta\mathscr{L}_2$ [9]. Pour cela il suffit de montrer qu'une boucle fermée d'où partent N photons et où arrive un photon de jauge est nulle sauf dans les cas où $N \leqslant 3$ pour lesquels interviennent les contre-termes contenus dans $\delta\mathscr{L}_2$.

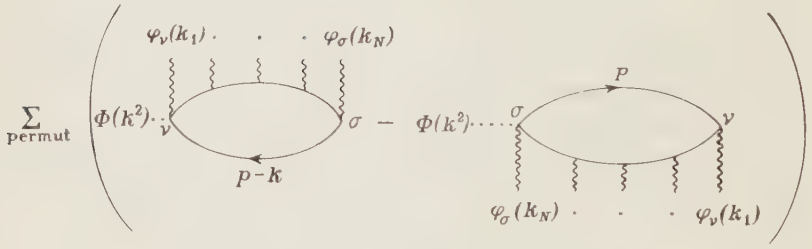
Considérons donc ce type de graphes, d'abord sans corrections radiatives:



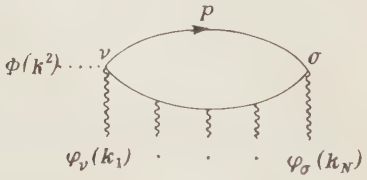
D'après

$$S(p-k)\gamma k\, S(p) = S(p-k) - S(p)$$

ce graphe devient



Par un déplacement de variables $p-k \to -p$ le premier graphe peut s'écrire:



mais parmi les différents graphes obtenus par permutation des lignes de photons, il y a toujours un graphe où l'ordre des lignes de photons $(k_1^\nu \dots k_N^\sigma)$ est inversé $(k_N^\sigma \dots k_1^\nu)$. Il y a alors compensation des graphes, si le déplacement de variables n'a pas introduit de « terme de surface », ce qui est le cas si $N > 3$.

Pour tenir compte des corrections radiatives on généralise ce calcul en utilisant la théorie de perturbation faite avec les constantes nues (e_0, m_0), ce qui évite l'inclusion dans les graphes des contre-termes de self-énergie.

Nous avons alors le même type de graphes que précédemment, à cette différence près, que certaines lignes de photons qui étaient ouvertes sont maintenant fermées. Le cas où le photon de jauge interagit avec les photons externes par l'intermédiaire de chaines de différentes boucles d'électrons reliées entre elles par des lignes de photons, n'a pas à être considéré, car, à de tels graphes, sont associés, dans la matrice S, les contre-termes convenables, qui sont tels que la boucle fermée où s'attache le photon de jauge soit nulle.

Comme dans la démonstration faite plus haut, nous pouvons passer du premier groupe de graphes au second en changeant, non seulement comme précédemment l'impulsion des électrons de $p - K$ en $-p$, mais aussi les impulsions des lignes internes de photons de l_i en $-l_i$.

Ce dernier changement étant sans effet sur les intégrales en l_i, le seul terme de surface ne peut venir que du déplacement de la variable d'impulsion de la ligne d'électron. Un tel terme n'est différent de zéro que si le nombre de nœuds sur cette ligne est $\leqslant 3$.

Pour $N = 3$ nous retrouvons ainsi le résultat que la seule constante λ_4^0 en α_0^2, suffit pour rendre invariante de jauge générale l'interaction $I_4^{\mu\nu\varrho\sigma}$.

Dans le cas $N = 2$, les graphes sont en principe nuls, par symétrie de charge: cependant il convient, pour être complet, de noter, que ce terme d'interaction de trois photons n'est pas nul lorsque on le calcule avec le seul procédé de symétrisation de Dirac-Heisenberg, — (il a alors la forme $\partial_\mu(\varphi_\mu\varphi^\nu\varphi_\nu)$ et est sans effet sur les équations du mouvement) — et qu'il est nécessaire de symétriser comme à la Sect. **3** pour obtenir le résultat voulu.

Le cas $N = 1$ peut se calculer directement.

Le tenseur de polarisation du vide le plus général est de la forme:

$$K^0_{\mu\lambda}(p) = \mathrm{Tr}\int \mathrm{d}^4k\gamma_\nu S'_{0F}(p+k)\Gamma^0_\mu(p+k,k)S'_{F0}(k), \tag{21}$$

$$= (p^2\delta_{\mu\nu} - p_\mu p_\nu)R^0(p^2) + \delta_{\mu\nu}Q^0(p^2)\,. \tag{22}$$

$Q_0(0)$ est relié à la constante μ_0^2 que nous avons déjà obtenue. Les autres termes du développement de

$$Q_0(p^2) = Q_0(0) + p^2\left(\frac{\partial Q_0}{\partial p^2}\right)_0 + \frac{(p^2)^2}{2!}\left(\frac{\partial^2 Q_0}{(\partial p^2)^2}\right)_0 + \dots,$$

peuvent s'obtenir en utilisant des relations déduites des identités de Ward généralisée,

$$-ik_\mu \Gamma_\mu(p, p+k) = S'^{-1}(p) - S'^{-1}(p+k) = -ik_\mu \Gamma_\mu(p+k, p)$$

(la dernière égalité étant obtenue en changeant k en $-k$, puis $p-k$ en p dans la première), et restreinte

$$i\Gamma_\mu(p, p) = \frac{\partial}{\partial p_\mu} S'^{-1}(p) .$$

Dérivant par rapport à k la première identité on a successivement:

$$\Gamma_\alpha(p, p+k) + k_\mu \frac{\partial}{\partial k_\alpha} \Gamma_\mu(p, p+k) = -i \frac{\partial}{\partial k_\alpha} S'^{-1}(p+k) ,$$

$$\frac{\partial}{\partial k_\beta} \Gamma_\alpha(p, p+k) + \frac{\partial}{\partial k_\alpha} \Gamma_\beta(p, p+k) + k_\mu \frac{\partial^2}{\partial k_\alpha \partial k_\beta} \Gamma_\mu(p, p+k) =$$

$$= -i \frac{\partial^2}{\partial k_\alpha \partial k_\beta} S'^{-1}(p+k) ,$$

et les mêmes équations avec $(p \rightleftarrows p+k)$.

Faisons tendre k vers zéro et introduisons la notation:

$$\left(\frac{\partial}{\partial k_\beta} \Gamma_\alpha(p, p+k) \right)_{k=0} = \Gamma_\alpha(p, \overline{p}_\beta) ,$$

on a:

$$\Gamma_\alpha(p, \overline{p}_\beta) + \Gamma_\beta(p, \overline{p}_\alpha) = -i \partial_{\alpha\beta} S_F^{p-1}(p) = \Gamma_\alpha(\overline{p}_\beta, p) + \Gamma_\beta(\overline{p}_\alpha, p)$$

mais

$$\frac{\partial \Gamma_\beta(p, p)}{\partial p_\alpha} = \Gamma_\beta(\overline{p}_\alpha, p) + \Gamma_\beta(p, \overline{p}_\alpha) = -i \partial_{\alpha\beta} S'^{-1}_F(p) ,$$

donc

$$\Gamma_\beta(p, \overline{p}_\alpha) = \Gamma_\alpha(\overline{p}_\beta, p) \tag{23}$$

et

$$\Gamma_\mu(\overline{p}_\mu, p) = \Gamma_\mu(p, \overline{p}_\mu) = \tfrac{1}{2} \partial_\mu \Gamma_\mu(p, p) \tag{24}$$

on a en généralisant la méthode

$$2\Gamma_\nu(\overline{p}_{\mu\nu}, p) + \Gamma_\mu(\overline{p}_{\nu\nu}, p) = \partial_\mu \partial_\nu \Gamma_\nu(p, p) . \tag{25}$$

De (22) on tire alors

$$(2\partial^\mu\partial^\nu K^0_{\mu\nu} + \partial_\mu\partial^\mu K^{0\nu}_{\nu})_{p^2=0} = 48\left(\frac{\partial Q_0(p^2)}{\partial p^2}\right)_{p^2=0},$$

terme qui détermine la constante ${}_0h_2$.

La même expression, calculée avec (21) en utilisant les relations (23), (24) et (25) conduit alors à l'expression

$$\sim \mathrm{Tr}\int \gamma_\nu \partial_\mu \partial^\mu \partial^\nu S'_{0F}\, \mathrm{d}^4k\,,$$

proportionnelle à la constante déjà calculée λ^0_4.

On trouve finalement que ${}_2h_0 - 1 = \alpha_0/3\pi$ et, de façon analogue, que le coefficient de $(\partial_\mu\varphi^\mu)\,\Box(\partial_\nu\varphi^\nu)$ relié à $(\partial^2\varphi/(\partial p^2)^2)_0$ est nul comme λ^0_6.

Les relations (23) à (25) ne permettent malheureusement pas d'obtenir, pour la partie de $K^0_{\mu\nu}(p^2)$ invariante de jauge une expression qui soit fonction du seul propagateur de l'électron. Les résultats ainsi obtenus par un calcul direct dans l'espace des impulsions, auraient pu être déduits, en généralisant la méthode de la Section **4**, introduisant dans le lagrangien la série des termes arbitraires $\sum_{i=0}^{\infty} h_i(\partial_\mu q'')^i$, $\sum \eta_i(\partial_\mu q'')\,\Box^i(\partial_\nu q'')$ et jouant sur la dépendance de jauge des constantes ${}_0h_i$ (en particulier de h_1 et h_0).

8. – Conclusion.

Nous avons montré que la contradiction entre le principe d'invariance de jauge et l'apparition d'une masse du photon nu est en fait levée lorsque, dans le lagrangien ou dans les équations des champs, l'on associe aux expressions faisant intervenir cette masse, celles dans lexquelles apparaissent les quantités non invariantes de jauge que sont la densité d'énergie du vide et la densité de courant du vide. Par ailleurs nous avons trouvé que la masse du photon n'est pas non plus invariante de jauge. Cette propriété résulte de l'introduction dans les équations du photon d'un terme d'interaction non linéaire photon-photon, nécessaire pour assurer la validité du théorème de Ward.

Il convient à ce sujet de souligner l'importance de ce terme de renormalisation de l'interaction photon-photon non invariante de jauge dont l'ommission, si elle n'a été compensée par une prescription convenable, peut introduire une divergence infrarouge dans le calcul de l'effet Delbrück.

La théorie ainsi construite a sur la théorie conventionnelle l'avantage de principe de relier les invariances pratiques de jauge de la matrice S à l'existence du groupe mathématique d'invariance de jauge des équations, évitant ainsi

de tronquer la matrice S pour obtenir les invariances pratiques que l'on est en droit d'attendre pour des raisons physiques.

Enfin « prenant au sérieux » un ensemble de termes que l'on considère d'habitude comme parasites, et qu'on s'ingénie à éliminer, nous trouvons au contraire que ces termes ont la propriété intéressante d'être les seuls que les méthodes actuellement connues permettent de calculer à tout ordre de la théorie de perturbation. Puisque ces termes sont, par les formules de la Section **5**, liés aux grandeurs observables, nous obtenons ainsi, certaines informations sur les grandeurs observables, et cela, à tout ordre de la théorie de perturbation. Ces résultats seront exposés dans un prochain article.

Cependant un mystère n'a pas été élucidé: l'électrodynamique quantique déduite des équations usuelles de Maxwell-Dirac, dans lesquelles le photon a une masse nue nulle et n'a pas de self couplage, ne semble pas pouvoir représenter le photon physique, aussi bien d'après la théorie de perturbation que d'après l'argument global de la Section **5**: il resterait cependant à voir comment la structure de cette théorie peut être telle qu'avec une masse de méson « observable » qui semble être infinie, et des interactions « observables » méson-méson de contact, une espèce d'invariance pratique de jauge puisse d'après les théorèmes de la Section **1** en émerger. Ces équation suggèrent évidemment une explication possible des propriétés encore incomprises des champs introduits par YANG et MILLS ([28]).

* * *

Nous tenons à remercier Monsieur R. STORA, pour d'intéressantes discussions.

([28]) C. N. YANG et R. L. MILLS: *Phys. Rev.*, **96**, 191 (1954); J. J. SAKURAI: *Ann. Phys.*, **11**, 1 (1960).

RIASSUNTO (*)

Prima si richiamano i principi della invarianza di gauge e studiano le violazioni che subiscono nella teoria della perturbazione: cioè, la ben nota autoenergia dei fotoni; ed il termine di interazione fotone-fotone comunemente trascurato. Poi si costruisce una nuova equazione del campo fotonico, che comporta nuove costanti di rinormalizzazione, tale che la matrice S è invariante di gauge, senza ricorrere alla regolarizzazione. L'invarianza di gauge delle equazioni di campo, fornisce le relazioni per le nuove costanti di rinormalizzazione. Il calcolo non perturbativo prova la consistenza di queste relazioni e connette le nuove costanti alle quantità osservabili. I valori di queste costanti vengono poi calcolati in ogni ordine della teoria della perturbazione.

(*) *Traduzione a cura della Redazione.*

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Dispersion Relations and the Causality Concept (*)(**).

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Summary. — It is well known that the scattering function $S(\varkappa)$ associated with a cut potential has certain analytic properties that make it satisfy dispersion relations. It is of interest to see how these analytic properties are modified when the potentials are not cut-off at a certain point, but continue to infinity, going asymptotically to zero there. The discussion is first carried using a causality condition enunciated as follows: The wave function associated with any initial wave packet remains bounded for all time. As a consequence of the causality condition, we obtained that it is no longer the $S(\varkappa)$, but a new function, which we call the dispersion function, that satisfies the analytic properties that imply dispersion relations. We also check these analytic properties directly from the Schrödinger equation. Finally, to discuss the significance of the poles of the dispersion and scattering functions, we analize in detail the scattering by the Eckart potential, obtaining the time dependent Green function in terms of basic interaction Green (BIG) functions associated with the poles of the dispersion function. From the behaviour of the BIG functions, as functions of time, we can also obtain restrictions on the analytic behaviour of the dispersion function.

1. – Introduction.

It is well known that the s-wave scattering [1-3] by a cut potential of radius a leads to a scattering function $S(\varkappa)$ that is an analytic function of $\varkappa$ in the upper half I_+ of the complex plane of the wave number $\varkappa$, except pos-

(*) Work supported by the Instituto Nacional de la Investigación Científica, México

(**) An abstract of this paper appears in the *Proceedings of the Tenth Annual International Conference on High Energy Physics.*

(1) J. Tiomno and W. Schützer: *Phys. Rev.*, **83**, 249 (1951).

(2) M. Moshinsky: *An. Acad. Brasileira Ciencias*, **4**, 343 (1952).

(3) A. Martin: *Nuovo Cimento*, **14**, 403 (1959).

sibly for poles on the positive imaginary axis that corresponds to bound states. Besides, an elementary discussion of the square well potential of range a shows that

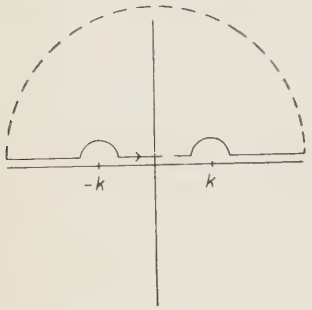

Fig. 1. – Integration contour in the $\varkappa$-plane.

$$S(\varkappa)\exp[i2\varkappa a]\to 0\,, \tag{1}$$

when $\varkappa$ is in I_+ and $|\varkappa|\to\infty$. A general proof of this property for an arbitrary cut potential of range a will be given in Section **3** using the Born approximation.

As a consequence of the properties of $S(\varkappa)$ given in the previous paragraph, we see that in the absence of bound states, the following integral vanishes

$$\int_C [2i(\varkappa^2-k^2)]^{-1}\{\exp[i\varkappa(r-r_0)]-S(\varkappa)\exp[i\varkappa(r+r_0)]\}\,\mathrm{d}\varkappa=0\,, \tag{2}$$

where C is the contour of Fig. 1, and $r\geqslant r_0\geqslant a$. The proof of (2) is immediate if we complete the contour C by the dotted contour of Fig. 1, and use (1) and the Cauchy theorem. In particular, if $r=r_0=a$, we get from (2) the dispersion relation

$$\mathrm{Im}[S(k)\exp[i2ka]]=-\frac{2k}{\pi}P\int_0^\infty \frac{\mathrm{Re}[S(\varkappa)\exp[i2\varkappa a]]}{\varkappa^2-k^2}\,\mathrm{d}\varkappa\,. \tag{3}$$

where Im, Re stands for imaginary and real part respectively, and P indicates the principal value.

For the non cut potentials, MARTIN [3] and others (*) have shown that the properties of $S(\varkappa)$ of the first paragraph no longer hold. For example, it is possible to construct potentials [3] with poles of $S(\varkappa)$ in I_- outside the imaginary axis, and also to have poles on the imaginary axis that do not correspond to bound states. It follows therefore that, in general, for non cut potentials, $S(\varkappa)$ no longer has the analytic properties that make it satisfy equation (2) or the dispersion relation (3).

We could ask ourselves what would be the generalization of equation (2) to arbitrary potentials, so as to obtain a new function, which we will denote by the name of dispersion function, that satisfies dispersion relations.

In Section **2** we shall obtain the dispersion function using essentially a causality principle formulated as follows: the time dependent Green function

(*) See the article of A. MARTIN for other references.

for the scattering by an arbitrary potential should be bounded for all times. As a consequence of this causality principle we shall see that the dispersion function is the Laplace transform of the time dependent Green function and that it satisfies an equation which reduces to (2) in the particular case of cut potentials of radius a.

In Section **3** we shall show directly that the dispersion function is analytic in I_+ except possibly for poles on the imaginary axis associated with bound states. Furthermore, using the Born approximation, we shall show that the dispersion function tends to zero as a function of $\varkappa$ when $\varkappa$ is in I_+ and $|\varkappa| \to \infty$. From these properties of the dispersion function, an equation that is a generalization of (2) follows immediately.

In Section **4** we discuss the particular case of scattering by the Eckart potential ([4,5]) which gives a very simple $S(\varkappa)$. We obtain explicitly the dispersion function and the time dependent Green function of this problem, and express the latter in terms of basic interaction Green (BIG) functions associated with the poles and zeros of $S(\varkappa)$. We shall show that the BIG functions of physical significance are those associated with the poles of $S(\varkappa)$ that are also poles of the dispersion function. From the asymptotic form of the BIG functions, when $t \to \infty$, we obtain restrictions on the analytic behaviour of the dispersion functions that are similar to those obtained in Section **2**. Using the properties of the BIG functions, we briefly discuss the significance of the complex poles and zeros of $S(\varkappa)$, comparing it with the results of our previous papers ([6,7]) and with the recent analysis of BECK and NUSSENZWEIG ([8]).

2. – Time dependent scattering by a potential and the causality principle.

In this section we shall give the time dependent description of the scattering by a potential, to be able to introduce a convenient definition of the causality principle.

For the sake of simplicity, we shall restrict ourselves to s-waves and a central potential, though the present discussion can be generalized to arbitrary angular momentum and non-central forces in an obvious way. The equation satisfied by the time dependent wave function $\varphi(r, t) \equiv r\psi(r, t)$ is then

$$\frac{1}{2}\frac{\partial^2\varphi(r,t)}{\partial r^2} - V(r)\,\varphi(r,t) = -i\frac{\partial\varphi(r,t)}{\partial t}\,, \tag{4}$$

([4]) C. ECKART: *Phys. Rev.*, **35**, 1303 (1930).
([5]) V. BARGMANN: *Rev. Mod. Phys.*, **21**, 488 (1949).
([6]) M. MOSHINSKY: *Phys. Rev.*, **84**, 525 (1951).
([7]) J. M. LOZANO: *Rev. Mexicana Fis.*, **2**, 155 (1953); **3**, 63 (1954).
([8]) G. BECK and M. NUSSENZWEIG: *Nuovo Cimento*, **16**, 416 (1960).

where $V(r)$ is the potential and $\hbar = m = 1$, m being the mass of the particle.

As the time dependent description of the scattering of a wave packet by a potential can be expressed by a superposition of Green functions, we shall restrict ourselves to the discussion of (4) with the initial condition

$$[\varphi(r, t)]_{t=0} = -(i/2) r_0^{-1} \delta(r - r_0) , \tag{5}$$

where r_0 is an arbitrary point. The time dependent Green function will be denoted by $\varphi(r, r_0, t)$.

To obtain $\varphi(r, r_0, t)$ we introduce its Laplace transform $\Phi(r, r_0, s)$ defined by

$$\Phi(r, r_0, s) = \int_0^\infty \varphi(r, r_0, t) \exp[-st] \, \mathrm{d}t . \tag{6}$$

As is well known, $\Phi(r, r_0, s)$ satisfies then the equation

$$\left[\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + is - V(r)\right] \Phi(r, r_0, s) = (2r_0)^{-1} \delta(r - r_0) , \tag{7}$$

and $\Phi(r, r_0, t)$ is given by [9]

$$\varphi(r, r_0, t) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Phi(r, r_0, s) \exp[st] \, \mathrm{d}s , \tag{8}$$

where c is a real positive constant such that

$$|\varphi(r, r_0, t)| < A \exp[ct] , \qquad \text{with } A > 0 \tag{9}$$

We shall now formulate the causality principle in the following way: the absolute value of the time dependent Green function should be bounded for all times (*). This formulation is physically obvious and it has also the advantage of not being related to any limiting velocity so that it is applicable to non-relativistic as well as relativistic problems. In the latter case, we must of course consider the $\varphi(r, r_0, t)$ associated with a Klein-Gordon or a Dirac equation.

[9] H. S. CARSLAW and J. C. JAEGER: *Operational Methods in Applied Mathematics* (Oxford, 1941), p. 72.

(*) This formulation of the causality principle does not hold for $t=0$ and $r=r_0$, but this is not relevant as the principle will hold if we average the Green function over any interval surrounding r_0.

From the causality condition enunciated in the previous paragraph, it follows that c could be taken as zero and from the theory of the Laplace transform ([9]), or directly from (6) we see that $\Phi(r, r_0, s)$ is an analytic function of s on the right side I_R of the plane s. Besides, $\Phi(r, r_0, s)$ is bounded in I_R as can be seen from the fact that

$$(10) \quad |\Phi(r, r_0, s)| = \left| \int_0^\infty q(r, r_0, t) \exp[-st]\, dt \right| \leqslant \int_0^\infty \exp[-s_x t]\, |q(r, r_0, t)|\, dt \leqslant (A/s_x),$$

where $s_x = \mathrm{Re}\, s$.

We now express s in terms of a new variable $\varkappa$ as

$$(11) \qquad s = -(i/2)\varkappa^2 .$$

The Laplace transform in terms of this new variable will be designated as $\overline{\varphi}(r, r_0, \varkappa)$ *i.e.*,

$$(12) \qquad \overline{\varphi}(r, r_0, \varkappa) \equiv \Phi(r, r_0, s) ,$$

and it satisfies the equation

$$(13) \qquad \left[\frac{d^2}{dr^2} + \varkappa^2 - 2V(r)\right] \overline{\varphi}(r, r_0, \varkappa) = r_0^{-1}\, \delta(r - r_0) .$$

The contour parallel to the imaginary axis in the plane of Fig. 2*A*, transforms into the contour in the $\varkappa$ plane of Fig. 2*B*. The analysis given above

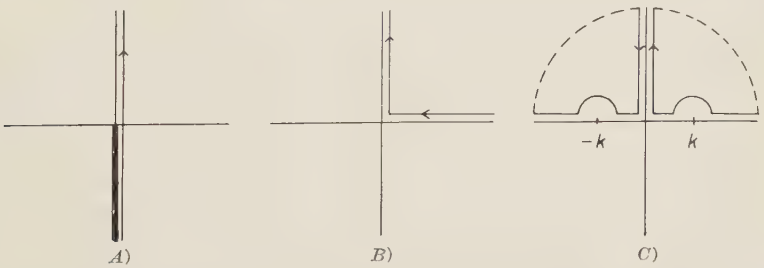

Fig. 2. – *A*) Integration contour for the time dependent Green function in the s-plane. *B*) Integration contour for the time dependent Green function in the $\varkappa$-plane. *C*) Integration contour for the dispersion function in the $\varkappa$-plane.

now indicates that $\overline{\varphi}(r, r_0, \varkappa)$ is analytic in the first quadrant of the plane $\varkappa$, and it tends to zero when $\varkappa$ is inside the first quadrant and $|\varkappa| \to \infty$, as in (10) we can substitute s_x by $(\varkappa_x \varkappa_y)$ where $\varkappa_x$, $\varkappa_y$ are respectively the real and imaginary parts of $\varkappa$. Furthermore, if the potential $V(r) \to 0$ ([5]) faster than r^{-1}

when $r \to -\infty$, the function $\overline{\varphi}(r, r_0, \varkappa)$ tends to a superposition of $\exp[\pm i\varkappa r]$ when $r \to -\infty$, but as from (10) $\overline{\varphi}(r, r_0, \varkappa)$ is bounded inside the first quadrant, we see that only the outgoing wave $\exp[i\varkappa r]$ can appear in $\varphi(r, r_0, \varkappa)$ when $r \to \infty$.

From the properties of the δ function, and the previous remarks, we see that $\overline{\varphi}(r, r_0, \varkappa)$ is completely defined by the following set of relations

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \varkappa^2 - 2V(r)\right]\overline{\varphi}(r, r_0, \varkappa) = 0 , \qquad \text{if } \begin{cases} 0 \leqslant r < r_0 \\ r_0 < r \end{cases} \tag{14a}$$

$$\overline{\varphi}(0, r_0, \varkappa) = 0 , \tag{14b}$$

$$\overline{\varphi}(r_0 + 0, r_0, \varkappa) = \overline{\varphi}(r_0 - 0, r_0, \varkappa) , \tag{14c}$$

$$\left[\frac{\mathrm{d}\overline{\varphi}(r, r_0, \varkappa)}{\mathrm{d}r}\right]_{r=r_0-0}^{r=r_0+0} = \frac{1}{r_0} , \tag{14d}$$

$$\lim_{r\to\infty} \overline{\varphi}(r, r_0, \varkappa) \propto \exp[i\varkappa r] . \tag{14e}$$

From (14) we see that the functions $[\overline{\varphi}(r, r_0, \varkappa)]^*$ and $\overline{\varphi}(r, r_0, -\varkappa^*)$ will satisfy exactly the same set of equations and therefore, we obtain the symmetry property

$$[\overline{\varphi}(r, r_0, \varkappa)]^* = \overline{\varphi}(r, r_0, -\varkappa^*) . \tag{15}$$

This implies, in particular, that the function $\overline{\varphi}(r, r_0, \varkappa)$ with $\varkappa$ in the second quadrant is just the conjugate of $\overline{\varphi}(r, r_0, -\varkappa^*)$, where now $-\varkappa^*$ is in the first quadrant.

The function $\overline{\varphi}(r, r_0, \varkappa)$ is then analytic in the upper half I_+ of the complex plane $\varkappa$ except possibly, for the positive imaginary axis. It also tends to zero when $\varkappa$ is in I_+ and $|\varkappa| \to \infty$, again with the possible exception of the positive imaginary axis. We shall show in the next section that the only singularities of $\overline{\varphi}(r, r_0, \varkappa)$ on the positive imaginary axis are poles and these are associated with bound states. If, for the sake of simplicity, we restrict ourselves to potentials without bound states, we could apply Cauchy theorem to the function $\overline{\varphi}(r, r_0, \varkappa)$ along the contour of Fig. 2*C* to obtain:

1) The function $\overline{\varphi}(r, r_0, \varkappa)$ satisfies the equation

$$\int_C (\varkappa^2 - k^2)^{-1} (\varkappa r_0)\, \overline{\varphi}(r, r_0, \varkappa)\, \mathrm{d}\varkappa = 0 , \tag{16}$$

where C is the contour of Fig. 1.

2) The time dependent Green function $\varphi(r, r_0, t)$ given by the inverse Laplace transform (8), becomes

$$\varphi(r, r_0, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\varphi}(r, r_0, \varkappa) \exp\left[-\frac{i}{2} \varkappa^2 t\right] \varkappa \mathrm{d}\varkappa , \tag{17}$$

where the integration is taken along the real axis in the $\varkappa$ plane.

If the potential is cut at the point a, and if we take $r \geqslant r_0 \geqslant a$, the equation (14) will give immediately that

$$\overline{\varphi}(r, r_0, \varkappa) = (2i\varkappa r_0)^{-1} \{\exp[i\varkappa(r - r_0)] - S(\varkappa) \exp[i\varkappa(r + r_0)]\} , \tag{18}$$

where $S(\varkappa)$ is the scattering function of the problem. The equation (16) becomes then precisely the equation (2) that leads to the dispersion relation (3) for the case of cut potentials.

We could consider then that (16) is the generalization of (2) for an arbitrary potential. We shall designate $\overline{\varphi}(r, r_0, \varkappa)$ as the dispersion function, as from the causality requirements it has the analytic properties that make it satisfy equation (16), and therefore lead to a dispersion relation.

In the following section we shall show directly, with the help of the JOST [10] functions, that $\overline{\varphi}(r, r_0, \varkappa)$ has the analytic properties obtained in this section with the help of the causality principle.

3. – Analytic properties of the dispersion function.

We shall start by defining the Jost [10] functions $f(\pm\varkappa, r)$ with the help of which we shall give the explicit form of the dispersion function $\overline{\varphi}(r, r_0, \varkappa)$. The functions $f(\pm\varkappa, r)$ satisfy the equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \varkappa^2 - 2V(r)\right] f(\pm\varkappa, r) = 0 , \qquad 0 \leqslant r \leqslant \infty , \tag{19}$$

and the asymptotic condition

$$\lim_{r \to \infty} f(\pm\varkappa, r) = \exp[\mp i\varkappa r] . \tag{20}$$

If the potential $V(r)$ satisfies

$$\int_0^{\infty} |V(r)| \mathrm{d}r < \infty , \tag{21}$$

[10] R. JOST: *Helv. Phys. Acta*, **20**, 256 (1947).

then $f(-\varkappa, r)$ as a function of $\varkappa$ is analytic in I_+ and $f(\varkappa, r)$ is analytic in I_-, as was shown by BARGMANN [5]. Besides, by an analysis entirely similar to the one that leads to (15), we can prove that

$$[f(\varkappa, r)]^* = f(-\varkappa^*, r) . \tag{22}$$

The Jost functions for $r = 0$ will be designed simply by

$$f(\varkappa, 0) = f(\varkappa) , \qquad f(-\varkappa, 0) = f(-\varkappa) . \tag{23}$$

The regular solution of (19) must be zero for $r = 0$, and therefore it can be written as

$$F(\varkappa, r) \equiv (2i\varkappa)^{-1}[f(-\varkappa) f(\varkappa, r) - f(\varkappa) f(-\varkappa, r)] . \tag{24}$$

From the properties of the Wronskian, we have that

$$[\mathrm{d}F(\varkappa, r)/\mathrm{d}r]_{r=0} = -1 \tag{25}$$

and therefore, from a well known theorem of differential equations [11], we have that $F(\varkappa, r)$ is an entire function of $\varkappa$.

If we take the limit when $r \to \infty$, we obtain from (25) and (20) that

$$\lim_{r\to\infty} F(\varkappa, r) = \frac{f(-\varkappa)}{2i\varkappa}\left[\exp[-i\varkappa r] - \frac{f(\varkappa)}{f(-\varkappa)} \exp[i\varkappa r]\right] , \tag{26}$$

so that the scattering function $S(\varkappa)$ is given by

$$S(\varkappa) = [f(\varkappa)/f(-\varkappa)] . \tag{27}$$

From the equation (14 a, b, e) it follows that

$$\begin{cases} \overline{\varphi}(r, r_0, \varkappa) = A(\varkappa) F(\varkappa, r) , & \text{if } 0 \leqslant r < r_0, \\ \overline{\varphi}(r, r_0, \varkappa) = B(\varkappa) f(-\varkappa, r) , & \text{if } r > r_0. \end{cases} \tag{28}$$

Using now the equations (14 c, d) we obtain

$$\overline{\varphi}(r, r_0, \varkappa) = r_0^{-1} F(\varkappa, r_0)[f(-\varkappa, r)/f(-\varkappa)] , \qquad \text{if } r \geqslant r_0. \tag{29}$$

From the properties of $f(-\varkappa, r)$ and $F(\varkappa, r_0)$ given in the previous paragraphs, it follows that $\overline{\varphi}(r, r_0, \varkappa)$ is analytic in I_+ of the $\varkappa$ plane, except pos-

[11] E. L. INCE: *Ordinary Differential Equations*, p. 72.

sibly at the zeros of $f(-\varkappa)$ in I_+. We shall prove that the zeros of $f(-\varkappa)$ are either in I_- or on the positive imaginary axis, and that in the latter case they correspond to bound states.

From the equations (19) satisfied by $f(-\varkappa, r)$ and $f(\varkappa^*, r)$, and from (22), we obtain

$$(30) \qquad \left\{[f(-\varkappa, r)]^* f'(-\varkappa, r) - f(-\varkappa, r) f'(\varkappa^*, r)\right\}_0^\infty + \\ + (\varkappa^2 - \varkappa^{*2}) \int_0^\infty |f(-\varkappa, r)|^2 \, \mathrm{d}r = 0 .$$

If $\varkappa$ is in I_+, *i.e.* $\varkappa = \varkappa_x + i\varkappa_y$ with $\varkappa_y > 0$, we have from (20) that the parenthesis in curly brackets in (30) vanishes when $r \to \infty$, so that using the notation (23) we obtain

$$(31) \qquad \left\{[f(-\varkappa)]^* f'(-\varkappa, 0) - f(-\varkappa) f'(\varkappa^*, 0)\right\} = (\varkappa^2 - \varkappa^{*2}) \int_0^\infty |f(-\varkappa, r)|^2 \, \mathrm{d}r .$$

If $\varkappa = \varkappa_0$ is a value in I_+ such that $f(-\varkappa_0) = 0$, which implies that $[f(-\varkappa_0)]^* = 0$, also, then

$$(32) \qquad (\varkappa_0^2 - \varkappa_0^{*2}) \int_0^\infty |f(-\varkappa_0, r)|^2 \, \mathrm{d}r = 0 ,$$

and therefore, $\varkappa_0$ can only be on the positive imaginary axis. The function $f(-\varkappa)$ cannot be zero for $\varkappa$ real, because from (22) we would have $[f(-\varkappa)]^* = f(\varkappa)$ for $\varkappa$ real, and therefore the Wronskian of $f(-\varkappa, r)$ and $f(\varkappa, r)$ would be zero for $r = 0$, which would lead to the contradiction that $f(\varkappa, r)$ and $f(-\varkappa, r)$ are not linearly independent.

For the zeros of $f(-\varkappa)$ on the positive imaginary axis, $\varkappa = i|\varkappa|$, and we have from (24) that

$$(33) \qquad F(i|\varkappa|, r) = \frac{f(i|\varkappa|)}{2|\varkappa|} f(-i|\varkappa|, r) ,$$

so that the regular solution of the equation (19) goes exponentially to zero for $r \to \infty$, so that it represents a bound state. We can therefore say that in the absence of bound states the zeros of $f(-\varkappa)$ are in I_-.

From the analysis of the previous paragraph we see that $\overline{\varphi}(r, r_0, \varkappa)$ is analytic in I_+ of the $\varkappa$ plane in the absence of bound states. The only remaining point is to prove that $\overline{\varphi}(r, r_0, \varkappa)$ tends to zero for $\varkappa$ in I_+ if $|\varkappa| \to \infty$. We

shall use the Born approximation, which is certainly valid when $|\varkappa| \to \infty$, to prove this point.

By direct substitution we can prove that the equations (14) are satisfied if $\overline{\varphi}(r, r_0, \varkappa)$ obeys the following integral equation [12]

$$(34) \qquad \overline{\varphi}(r, r_0, \varkappa) = -\sin(\varkappa r)\int_r^\infty \varkappa^{-1} \exp[i\varkappa x]\, 2V(x)\, \overline{\varphi}(x, r_0, \varkappa)\, \mathrm{d}x - \\ - \varkappa^{-1} \exp[i\varkappa r]\int_0^r \sin(\varkappa x)\, 2V(x)\, \overline{\varphi}(x, r_0, \varkappa)\, \mathrm{d}x + u(r, r_0)\,,$$

where

$$(35a) \qquad u(r, r_0) = -(\varkappa r_0)^{-1} \exp[i\varkappa r] \sin(\varkappa r_0)\,, \qquad r \geqslant r_0.$$

For $r \leqslant r_0$, the integral equation is similar to (34) but only replacing $u(r, r_0)$ by

$$(35b) \qquad v(r, r_0) = -(\varkappa r_0)^{-1} \exp[i\varkappa r_0] \sin(\varkappa r)\,, \qquad r \leqslant r_0.$$

In the absence of a potential, $\overline{\varphi}(r, r_0, \varkappa)$ reduces to $u(r, r_0)$ if $r \geqslant r_0$ or to $v(r, r_0)$ if $r \leqslant r_0$. Therefore, in the first Born approximation, we can replace the $\overline{\varphi}(x, r_0, \varkappa)$ that appears under the integral sign in (34) by $u(x, r_0)$ if $x \geqslant r_0$ or $v(x, r_0)$ if $x \leqslant r_0$, and we obtain for $r_0 \leqslant r$,

$$(36) \qquad \overline{\varphi}(r, r_0, \varkappa) \simeq \varkappa^{-1} \sin(\varkappa r)\int_r^\infty 2V(x) \exp[i\varkappa x](\varkappa r_0)^{-1} \exp[i\varkappa x] \sin(\varkappa r_0)\, \mathrm{d}x + \\ + \varkappa^{-1} \exp[i\varkappa r]\int_{r_0}^r 2V(x) \sin(\varkappa x)(\varkappa r_0)^{-1} \exp[i\varkappa x] \sin(\varkappa r_0)\, \mathrm{d}x + \\ + \varkappa^{-1} \exp[i\varkappa r]\int_0^{r_0} 2V(x) \sin(\varkappa x)(\varkappa r_0)^{-1} \sin(\varkappa x) \exp[i\varkappa r_0]\, \mathrm{d}x + u(r, r_0)\,,$$

and a similar expression for $r \leqslant r_0$.

All the integrals in (36) are of the type

$$(37) \qquad (2\varkappa^2 r_0)^{-1}\int V(x) \exp[i\varkappa\xi]\, \mathrm{d}x\,,$$

[12] N. F. MOTT and H. S. W. MASSEY: *Theory of Atomic Collisions* (Oxford, 1949), second edition, p. 108.

where ξ is a linear function of x that is always positive in the corresponding integration intervals. From (37), (21) and (35) it follows that $\overline{\varphi}(r, r_0, \varkappa)$ tends to zero when $\varkappa$ is in I_+ and $|\varkappa| \to \infty$, which was the point we wanted to prove. In particular, if we consider a cut potential of range a, we have that for $r \geqslant r_0 \geqslant a$, $\overline{\varphi}(r, r_0, \varkappa)$ has the form (18). Taking then the lower bound $r = r_0 = a$, we obtain from (18) and (36) that

$$(38) \qquad (2i\varkappa a)^{-1}\{1 - S(\varkappa)\exp[i2\varkappa a]\} \simeq -(2i\varkappa a)^{-1}\{\exp[i2\varkappa a] - 1\} + \mathcal{O}(\varkappa^{-2}),$$

where the first term on the right hand side corresponds to $u(a, a)$, while the second represents all the integrals in (36), which from (37) are at least of order $\varkappa^{-2}$. From (38) we see that the asymptotic condition (1) is satisfied if $\varkappa$ is in I_+ and $|\varkappa| \to \infty$.

We have shown that in the absence of bound states (*) $\overline{\varphi}(r, r_0, \varkappa)$ is an analytic function of $\varkappa$ in I_+ whose absolute value tends to zero when $|\varkappa \to \infty$ in I_+. Therefore, the equation (16), which was derived in the previous section using a causality principle, can be obtained directly from the analytic properties of the dispersion function defined by (14). It follows also that the time dependent Green function is determined by (17).

In the following section we shall obtain the explicit form of the time dependent Green function for the Eckart potential, for which the dispersion function $\overline{\varphi}(r, r_0, \varkappa)$ is particularly simple.

4. – The basic interaction Green (BIG) functions.

In this section we shall show that for the Eckart potential the time dependent Green function can be expressed in terms of basic interaction Green (BIG) functions that are associated with the poles and zeros of the scattering function $S(\varkappa)$.

We shall start by defining the Eckart [4] potential $V_E(r)$ and its corresponding Jost functions, as is done by BARGMANN [5]. The potential $V_E(r)$ is given by

$$(39) \qquad V_E(r) = -4\gamma\lambda^2 \frac{\exp[-2\lambda r]}{[\gamma \exp[-2\lambda r] + 1]^2},$$

(*) The presence of bound states can be easily taken into account by surrounding the corresponding poles in the positive imaginary axis by an appropriate contour, as suggested by Fig. 2C.

where (*) $-1<\gamma<1$, $\lambda>0$. The Jost functions associated with the Eckart potential are given by [5]

$$f(\pm\varkappa, r) = \exp[\mp i\varkappa r]\left[1 + i\,\frac{w(r)}{\pm\varkappa - i\lambda}\right], \tag{40a}$$

where

$$w(r) = \frac{2\gamma\lambda\exp[-2\lambda r]}{\gamma\exp[-2\lambda r]+1}. \tag{40b}$$

From (40) we see that

$$f(\pm\varkappa) = \frac{\varkappa \pm i\nu}{\varkappa \mp i\lambda}, \qquad S(\varkappa) = \frac{(\varkappa+i\nu)(\varkappa+i\lambda)}{(\varkappa-i\nu)(\varkappa-i\lambda)}, \tag{41}$$

where

$$\nu = \lambda[(\gamma-1)/(\gamma+1)]. \tag{42}$$

From the expression (24), (29) for the dispersion function $\overline{\varphi}(r, r_0, \varkappa)$, we obtain for the Eckart potential

$$\begin{aligned}\overline{\varphi}(r, r_0, \varkappa) = \\ = \frac{1}{2i\varkappa r_0}\left\{-2i\sin(\varkappa r_0) - 2\,\frac{\exp[\lambda r_0]\,w(r_0)}{\gamma+1}\left[\frac{\sin(\varkappa+i\lambda)r_0}{\varkappa+i\lambda} - \frac{\sin(\varkappa-i\lambda)r_0}{\varkappa-i\lambda}\right]\right\}\cdot \\ \cdot\left[1 + i\,\frac{\lambda+\nu-w(r)}{\varkappa-i\nu}\right]\exp[i\varkappa r].\end{aligned} \tag{43}$$

Substituting (43) into (17), and expressing the sines in terms of exponentials, we see that the integrals to be evaluated are of two types

$$G(r\pm r_0, t) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\exp\left[i\left|\varkappa(r\pm r_0) - \frac{1}{2}\varkappa^2 t\right|\right]d\varkappa, \tag{44}$$

$$\chi(r\pm r_0, k, t) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\exp\left[i[\varkappa(r\pm r_0) - \frac{1}{2}\varkappa^2 t]\right]}{\varkappa - k}\,d\varkappa. \tag{45}$$

The function $G(r\pm r_0, t)$ is the time dependent Green function of the problem in the absence of a potential which, as is well known, is given by

$$G(r, t) = -(2/\pi t)^{\frac{1}{2}}\exp[i\pi/4]\exp[ir^2/2t]. \tag{46}$$

(*) If $\gamma>1$, we could have a bound state, and the following analysis would have to be modified slightly.

The function $\chi(r \pm r_0, k, t)$ appears associated with those values $\varkappa = -k$ that are the zeros and poles of $f(-\varkappa)$ and poles of $f(\varkappa)$, which for the Eckart potential mean the two poles and one of the zeros of $S(\varkappa)$. We shall designate the χ functions by the name of basic interaction Green (BIG) functions, as they are associated with the intrinsic properties (zeros and poles) of the scattering function $S(\varkappa)$, a function which, in turn, describes the interaction process. The functions χ have been discussed previously ([6]), and their explicit form is given by

$$\chi(r, k, t) = \exp\left[i(kr - \tfrac{1}{2}k^2 t)\right] \operatorname{erfc}\left[(1-i)(r-kt)(4t)^{-\frac{1}{2}}\right], \tag{47}$$

where

$$\operatorname{erfc}(z) = 2\pi^{-\frac{1}{2}} \int_z^\infty \exp\left[-u^2\right] \mathrm{d}u\,. \tag{48}$$

From the properties of erfc (z) we obtained ([6]) the following asymptotic form for the χ's when $t \to \infty$,

$$\lim_{t\to\infty} \chi(r, k, t) = \begin{cases} -(2/\pi k^2 t)^{\frac{1}{2}} \exp i\left[\dfrac{\pi}{4}\right] \exp\left[ir^2/2t\right], & \dfrac{3}{4}\pi < \arg k < \dfrac{7\pi}{4}, \\ 2\exp\left[i(kr - \tfrac{1}{2}k^2 t)\right] - \\ -(2/\pi k^2 t)^{\frac{1}{2}} \exp\left[i\,\dfrac{\pi}{4}\right] \exp\left[ir^2/2t\right], & -\dfrac{\pi}{4} < \arg k < \dfrac{3\pi}{4}. \end{cases} \tag{49}$$

From (17) and (43), (44), (45) we obtain finally that the time dependent Green function for the Eckart potential is

$$\begin{aligned} \varphi(r, r_0, t) = &-\frac{1}{4r_0}\left[G(r+r_0, t) - G(r-r_0, t)\right] + \\ &+ \frac{i}{4r_0}\,\frac{w(r_0)\,w(r)\exp\left[\lambda(r+r_0)\right]}{2\gamma\lambda} \cdot \\ &\cdot\left\{\exp\left[-\lambda(r+r_0)\right]\chi(r+r_0, -i\lambda, t) - \exp\left[-\lambda(r-r_0)\right]\chi(r-r_0, -i\lambda, t)\right\} - \\ &- \frac{i}{4r_0}\,\frac{w(r_0)\,w(r)\exp\left[\lambda(r+r_0)\right]}{2\gamma\lambda} \cdot \\ &\cdot\left\{\exp\left[\lambda(r+r_0)\right]\chi(r+r_0, i\lambda, t) - \exp\left[\lambda(r-r_0)\right]\chi(r-r_0, i\lambda, t)\right\} - \\ &- \frac{i}{4r_0}\,\frac{(1-\gamma^2)}{2\gamma\lambda}\left[\lambda+\nu-w(r_0)\right]\left[\lambda+\nu-w(r)\right]\chi(r+r_0, i\nu, t)\,. \end{aligned} \tag{50}$$

From (50) we see that the time dependent Green function can be expressed in terms of the functions $\chi(r \pm r_0, k, t)$ for those values of k that are zeros

of $f(-\varkappa)$, poles of $f(-\varkappa)$ and poles of $f(\varkappa)$. These values of k are going to be designated by $\times$, $\circ$ and $\otimes$ respectively, and for the Eckart potential are given in Fig. 3*A*.

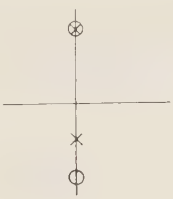

Fig. 3*A*. – Eckart potential.

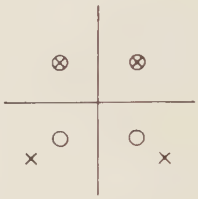

Fig. 3*B*. – Bargmann potential.

$\times$ zeros of $f(-\varkappa)$, $\circ$ poles of $f(-\varkappa)$, $\otimes$ poles of $f(\varkappa)$.

It is interesting to note from (50) that the $\chi(r \pm r_0, k, t)$ for the k associated with $\circ$ and $\otimes$ of Fig. 3*A*, appear combined in such a way, that the combinations tends to zero when $t \to \infty$, as can be seen immediately using (49). This behaviour is true even for more complex potentials. For example BARGMANN ([13]), using the methods of Jost and Kohn ([14]), and of Gelfand and Levitan ([15]), has constructed a potential for which the $f(\pm\varkappa)$ is given by

$$f(\pm\varkappa) = \frac{(\pm\varkappa - i\alpha)(\pm\varkappa - i\alpha^*)}{(\pm\varkappa - i\beta)(\pm\varkappa - i\beta^*)}\,, \tag{51}$$

where α, β are complex numbers whose real and imaginary parts are positive. The $\chi(r \pm r_0, k, t)$, that appear in the solution of this problem, correspond to the values of k

$$k = -i\alpha\,, \quad -i\alpha^*\,, \quad \pm i\beta\,, \quad \pm i\beta^*\,, \tag{52}$$

and are shown in Fig. 3*B*, where we follow the same convention as in Fig. 3*A*. From (49) the χ's associated with $k = i\beta^*$ will tend to infinity when $t \to \infty$, and this would appear to violate the causality principle. But the χ's appear combined in this problem ([16]) in the same way as in the curly brackets of (50), and therefore, their asymptotic forms cancel and there is no violation of the causality principle, as we would of course expect.

([13]) V. BARGMANN: private communication.

([14]) K. JOST and W. KOHN: *Mat. Fys. Medd.*, **27**, (9) (1953).

([15]) I. M. GELFAND and B. M. LEVITAN: *Amer. Math. Soc. Trans.*, Sec, **21**, 250 (1955).

([16]) J. M. LOZANO: *Rev. Mexicana Fis.*, **9**, 149 (1960).

The discussion of the previous paragraph seems to suggest that the BIG functions $\chi(r \pm r_0, k, t)$ associated with values k of the type ○ and ◌ are not physically relevant, and rather represent a kind of geometrical effect associated with the source point r_0. This can be seen more clearly if we pass to the limit $r_0 \to 0$. From the properties of the Wronskian, the function $r_0^{-1}F(\varkappa, r_0)$ of (24) tends to -1 when $r_0 \to 0$, so that $\overline{\varphi}(r, 0, \varkappa)$ reduces to

$$\overline{\varphi}(r, 0, \varkappa) = -\frac{f(-\varkappa, r)}{f(-\varkappa)} . \tag{53}$$

Using (17) and (40), (41), we see that $\varphi(r, 0, t)$, for the case of the Eckart potential becomes

$$\varphi(r, 0, t) = \tfrac{1}{2}\left[\lambda + \nu - w(r) - i(2r/t)\right] G(r, t) + \frac{i\nu}{2}\left[\lambda + \nu - w(r)\right] \chi(r, i\nu, t), \tag{54}$$

which shows that for $r_0 = 0$, we only have BIG functions associated with k's of the type ×, that is, with zeros of $f(-\varkappa)$.

In general, we shall show that $\overline{q}(r, 0, \varkappa)$ of (53) has only poles at the zeros of $f(-\varkappa)$. We already showed in the previous section that $\overline{\varphi}(r, r_0, \varkappa)$ is analytic in I_+ of the $\varkappa$ plane in the absence of bound states. We need then only concern ourselves with the behaviour of $\overline{\varphi}(r, 0, \varkappa)$ in I_-. From (24) we have that

$$\frac{2i\varkappa F(\varkappa, r)}{f(-\varkappa) f(\varkappa)} = \frac{f(\varkappa, r)}{f(\varkappa)} - \frac{f(-\varkappa, r)}{f(-\varkappa)} , \tag{55}$$

where $F(\varkappa, r)$ is an entire function of $\varkappa$. Furthermore $f(-\varkappa, r)$ is analytic in in I_+, $f(\varkappa, r)$ is analytic in I_-, and in the absence of bound states, (that is, if $f(-i|\varkappa|) \neq 0$), $f(-\varkappa)$ has zeros only in I_- and $f(k)$ only in I_+. Therefore, $[f(\varkappa, r)/f(\varkappa)]$ is an analytic function in I_-, and $[2i\varkappa F(\varkappa, r)/f(\varkappa) f(-\varkappa)]$ has poles in I_- only at the zeros of $f(-\varkappa)$. From (55), (53) we conclude that the poles of $\overline{\varphi}(r, 0, k)$ are due only to the zeros of $f(-\varkappa)$. If instead of (21), $V(r)$ satisfies the stronger condition [5]

$$\int_0^\infty |V(r)| \exp\left[\alpha r\right] \mathrm{d}r < \infty \tag{56}$$

for any $\alpha > 0$, then the corresponding $f(-\varkappa, r)$ is analytic in the whole complex plane so that the only singularities of $\overline{q}(r, 0, \varkappa)$ will be poles. Developing $\overline{q}(r, 0, \varkappa)$ in a Mittag-Leffler series around these poles [17] the corresponding

[17] R. G. Newton: *Journ. Math. Phys.*, **1**, 319 (1960).

$q(r, 0, t)$ will contain BIG functions $\chi(r, k, t)$ associated only with k's of the type .

Another way of understanding the significance of the values k of the type ○ and ⊗, is to consider that as $F(\varkappa, r_0)$ is an entire function of $\varkappa$, we could develop it as

$$F(\varkappa, r_0) = F(k, r_0) + \left(\frac{\partial F}{\partial \varkappa}\right)_{\varkappa = k} (\varkappa - k) + \dots . \tag{57}$$

Taking for k values such that $f(-k) = 0$, we see for example, for the Eckart potential, that the time dependent Green function is a combination of functions $\chi(r \pm r_0, k, t)$ where k is of type ×, and of ordinary Green functions $G(r \pm r_0, t)$ and their derivatives with respect to r. From here we can conclude that the BIG functions associated with k's of the type ○ and ⊗ represent in essence the effect of translating the source point from the origin to r_0, and therefore these BIG functions correspond essentially to a kind of multipole development.

The BIG functions associated with k's of the type ×, *i.e.* with poles of the dispersion function $q(r, r_0, \varkappa)$ do have a physical significance, as they appear even when $r_0 = 0$. Besides, from (50) we see that their combination does not cancel in general when $t \to \infty$. In the case of the Bargmann potential, the dispersion function has a pole $k = -i\alpha$ (assuming $\alpha = \alpha_x + i\alpha_y$ with $0 < \alpha_x < \alpha_y$), and the corresponding $\chi(r, -i\alpha, t)$ has, from (49), the asymptotic behaviour

$$\chi(r, -i\alpha, t) \to 2 \exp\left[\alpha r + \frac{i}{2}\alpha^2 t\right] - (2/\pi k^2 t)^{\frac{1}{2}} \exp\left|i\frac{\pi}{4}\right| \exp\left[ir^2/2t\right], \tag{58}$$

so that besides a diffusion term, it has a term that decays exponentially with time and can be interpreted as a quasi-stationary state [6-8].

The correlation of the physically significant BIG functions that appear in the time dependent Green function, with those values of k associated exclusively with the zeros of $f(-\varkappa)$ and therefore with the poles of the dispersion function, allow us to formulate the causality concept in terms of the behaviour of the BIG function. From the asymptotic behaviour (49), we see that if k is in the first quadrant of the $\varkappa$ plane, then $\chi(r, k, t)$ tends to infinity when $t \to \infty$ and therefore, the causality condition discussed in Section 2 is not satisfied. This would indicate that the dispersion function $\bar{q}(r, r_0, \varkappa)$ can not have poles in the first quadrant of the $\varkappa$ plane, and because of the symmetry conditions (15), it can not have poles in the second quadrant either. Therefore, in the absence of bound states, $\bar{q}(r, r_0, \varkappa)$ cannot have poles in I_+. If, besides, $\bar{q}(r, r_0, \varkappa)$ satisfies the behaviour at infinity implied by (10), the dispersion relation implicit in (16) follows.

5. – Conclusion.

In this paper we have enunciated the causality condition in the following form: The time dependent Green function associated with the problem should be bounded for all times. From this causality condition one concludes the existence of a dispersion function, the Laplace transform of the time dependent Green function, that satisfies dispersion relations. For certain types of potential the time dependent Green function can be expressed in terms of basic interaction Green (BIG) functions associated with the poles of the dispersion functions. From the behaviour of the BIG functions as functions of time, one can obtain information on the analytic behaviour of the dispersion function in I_+.

In this paper we make use only of the BIG functions associated with the Schrödinger equation. We have also obtained [18] the BIG functions for the Klein-Gordon and Dirac equations. In future publications we plan to apply these BIG functions to the generalization of the present analysis to relativistic problems.

* * *

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[18] M. Moshinsky: *Symposium on New Techniques in Physics*, Rio de Janeiro, (1954), p. 285.

RIASSUNTO (*)

È ben noto che la funzione di scattering $S(\varkappa)$ associata ad un potenziale di taglio ha certe proprietà analitiche che fanno sì che essa soddisfi alle relazioni di dispersione. È interessante studiare come vengono modificate queste proprietà analitiche quando i potenziali non vengono tagliati a un determinato punto, ma continuano all'infinito, ivi tendendo asintoticamente allo zero. La discussione si fa dapprima usando una condizione di causalità enunciata come segue. La funzione d'onda associata ad ogni pacchetto d'onde iniziale rimane sempre limitata. In conseguenza della condizione di causalità, si è ottenuto che non è più la funzione $S(\varkappa)$, ma una nuova funzione, che chiamiamo funzione di dispersione, a soddisfare le proprietà analitiche che implicano relazioni di dispersione. Noi verifichiamo queste proprietà analitiche anche direttamente dalla equazione di Schrödinger. Infine, per discutere il significato dei poli delle funzioni di dispersione e scattering, analizziamo dettagliatamente lo scattering da parte del potenziale di Eckart, ottenendo la funzione di Green dipendente dal tempo in termini delle funzioni di interazione di Green (BIG) associate ai poli della funzione di dispersione. Dal comportamento delle funzioni BIG, in funzione del tempo, otteniamo anche le restrizioni al comportamento analitico della funzione di dispersione.

(*) *Traduzione a cura della Redazione.*

On the Decay Scheme of ^{128}Cs.

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Tata Institute of Fundamental Research - Bombay

(ricevuto il 2 Dicembre 1960)

Summary. — The results of the β-ray spectrometer and scintillation spectrometer studies of 2.5 min ^{128}Cs decay are presented. The Fermi-plot of the positron spectrum taken in Siegbahn-Slätis spectrometer gives the following end-point energies: (2885 ± 25) keV (100, log *ft* 4.8), (2445 ± 25) keV (39, log *ft* 5.1), (1900 ± 40) keV (⩽16, log *ft* ⩾5.1) and (1300 ± 40) keV (⩽8, log *ft* ⩾4.6). The analysis of the γ-ray spectra in scintillation spectrometers shows the following γ-rays: 168 keV (10), 270 keV (23), 440 keV (50), 511 keV (200), 970 keV (1.5), 1120 keV (2.1), 1660 keV (0.4), 2180 keV (0.36), 2420 keV (0.2). With the help of these data, β^+-γ and γ-γ coincidence studies, the following levels in ^{128}Xe are suggested: Ground state (0^+), 440 keV (2^+), 970 keV (2^+), 1560 keV (0^+ or 2^+) (all fed by positrons and electron captures), 2620 keV (perhaps 3^-) and 2860 keV (2^+) fed by electron captures only. It is suspected that the 2620 keV state may be the first octupole vibrational level.

1. – Introduction.

It is now well-established that the ground-state and the first excited state of near-spherical even-even nuclei are usually 0^+ and 2^+ respectively. The second excited state of such nuclei, if supposed to arise from the anharmonic surface oscillations, should be a triplet 0^+, 2^+, 4^+ ([1-4]). All the members of such a triplet have been observed in ^{114}Cd only. Normally, all the members of the triplet cannot be fed in β-decay because of the β-decay selection rules. But

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([2]) L. Wilets and M. Jean: *Phys. Rev.*, **102**, 788 (1956).
([3]) T. Tamura and L. C. Komai: *Phys. Rev. Lett.*, **3**, 344 (1959).
([4]) R. K. Sheline: *Rev. Mod. Phys.*, **32**, 1 (1960).

wherever the parent spin is $1^{\pm}$ or $2^{\pm}$ and enough decay energy is available, there is no apparent reason why the 0^{+} member of the second vibrational state should not be observed unless the energy separation of the 0^{+} and 2^{+} states is so small that radiations to and from them cannot be resolved. It was with the expectation of being able to throw some light on this aspect of the problem that the study of the decay of ^{128}Cs was undertaken.

The decay properties of 2.5 min ^{128}Cs are not known very well, mainly because of the difficulty of producing this isotope. All that is known is that it decays by the emission of positrons, the spectrum of which can be decomposed into three groups (5) with end-point energies at 3.0 MeV, 2.5 MeV and 1.5 MeV. Fink and Wiig (6) studied the mixture $^{128}Ba \rightarrow {}^{128}Cs \rightarrow {}^{128}Xe$ and found 135 keV, 290 keV, 460 keV, and 970 keV γ-rays. Hollander and Kalkstein (5) found a 270 keV γ-ray which they attributed to the decay of ^{128}Ba, and 440 keV and 980 keV γ-rays to ^{128}Cs. Wapstra, Verster and Boelhouwer (7) found 460 keV and 1500 keV γ-rays in the decay of ^{128}Cs. Novakov, Koički, Mladjenovič and Arbman (8) produced ^{128}Ba by (p, 6n) reaction by bombarding CsCl with 80 MeV protons in the Uppsala synchrocyclotron. From their study in the β-ray spectrometer, they found conversion lines which could be the K line of 162 keV γ-ray or the L line of 133.9 keV γ-ray, the K line of 277.7 keV γ-ray, K and L lines of 441.5 keV γ-ray and the K lines of 527.8 keV and 576.3 keV γ-rays. In addition, they found strong conversion lines of a 276 keV γ-ray, which they attributed to $^{143}Ba^{m}$. They found some additional conversion lines which could be identified as belonging to the decay of ^{131}Ba. In a scintillation spectrometer, they found 270 keV, 440 keV, 511 keV annihilation radiation, 940 keV and 1200 keV γ-rays. What intrigued us from the survey of these results was the absence of a positron group to the 970 keV second excited state of ^{128}Xe (9) and that although about 4 MeV decay energy is available, no higher energy γ-rays had been reported.

2. – Source preparation.

One gram of CsCl was bombarded for two hours with about 90 MeV protons in the Harwell cyclotron and the target was flown to Bombay. The Ba activity was co-precipitated on $Pb(NO_3)_2$ by fuming HNO_3. After two precipitations of $Pb(NO_3)_2$, the Ba activity was obtained carrier-free by removing

(5) J. M. Hollander and M. I. Kalkstein: *Phys. Rev.*, **98**, 260 (1955).
(6) F. W. Fink and E. O. Wiig: *Phys. Rev.*, **91**, 194 (1953).
(7) A. H. Wapstra, N. F. Verster and M. Boelhouwer: *Physica*, **19**, 138 (1953).
(8) T. Navakov, S. Koički, M. Mladjenovič and E. Arbman: *Comptes Rendus du Congres International de Physique Nucléaire* (Paris, 1959), p. 935.
(9) M. Sakai: *Institute of Nuclear Study, University of Tokyo, INSJ* 19 (1959).

Pb by precipitation with H_2S. From the supernate, the Ba activity was deposited on the source-holder by simple evaporation. The Ba activity, as later studies revealed, consisted of 39 hour $^{133}Ba^{m}$, 12 day ^{131}Ba and 2.4 day ^{128}Ba. The 2.5 min ^{128}Cs, which decays by positron emission and by electron captures to ^{128}Xe was in equilibrium with ^{128}Ba. The ^{128}Ba decays only by electron-capture to ^{128}Cs. The presence of ^{128}Ba did not interfere in the study of the positrons and the high energy γ-rays of ^{128}Cs.

3. – Beta-ray spectrometer studies.

The spectrum of positrons was measured in the intermediate focussing Siegbahn Slätis β-ray spectrometer, set for a resolution of 3.5%. The source was deposited on a (500 μ g/cm^2) mylar film; the solid matter in the source was less than one mg/cm^2. The Fermi plot of the spectrum is shown in Fig. 1.

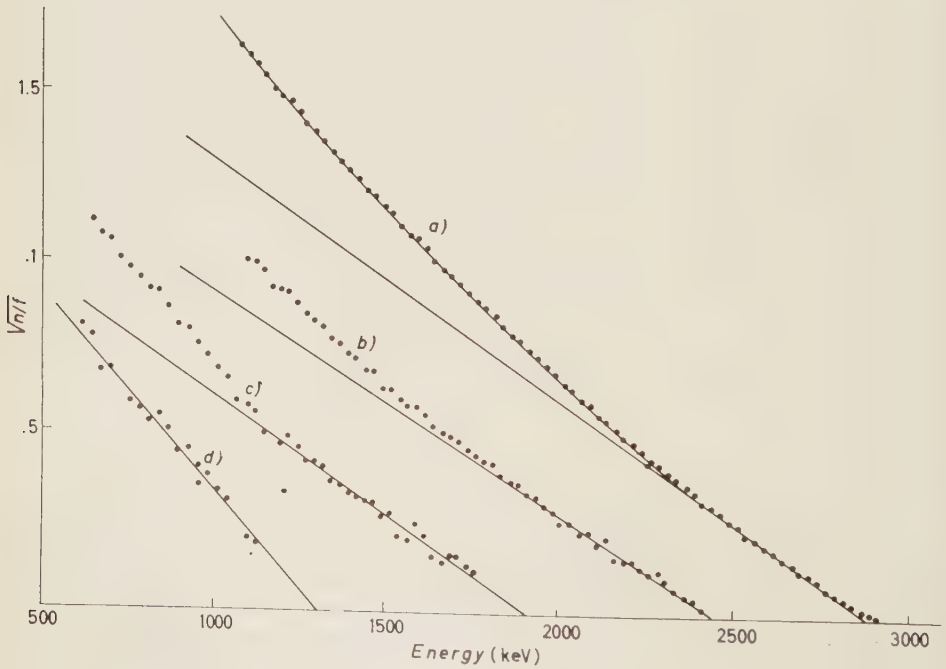

Fig. 1. – Fermi plot of the positron spectrum taken in Siegbahn-Slätis β-ray spectrometer.

The positron spectrum could be resolved into four groups with end-point energies and the intensities given in the Table I. The table also gives theoretical branching ratio (ε/β^+), for each group, of the K capture and positron decay.

Since the points for the two lower groups are scattered rather widely, the intensity values for these groups of positrons are the upper limits.

TABLE I.

End-point energy in keV	Relative intensity	ε/β^-	log *ft*
2 885 ± 25	100	0.35	4.8
2 445 ± 25	39	0.70	5.1
1 900 ± 40	16	1.2	5.1
1 300 ± 40	< 8	3.5	4.6

The low energy spectrum was studied with a thinner source. It was realized that for the identification of the conversion lines belonging to ^{128}Ba, one would have to study the decay of the intensities of these lines to rule out those belonging to ^{131}Ba and ^{133}Ba. This could not be done. But the occurrence of the conversion lines due to γ-rays of 270 keV and 168 keV could be confirmed.

4. – Scintillation spectrometer studies.

The spectrum of γ-rays from the mixture ^{128}Ba and ^{128}Cs was studied in a scintillation spectrometer, with a NaI(Tl) crystal $1\frac{1}{2}$ in. diameter and 2 in.

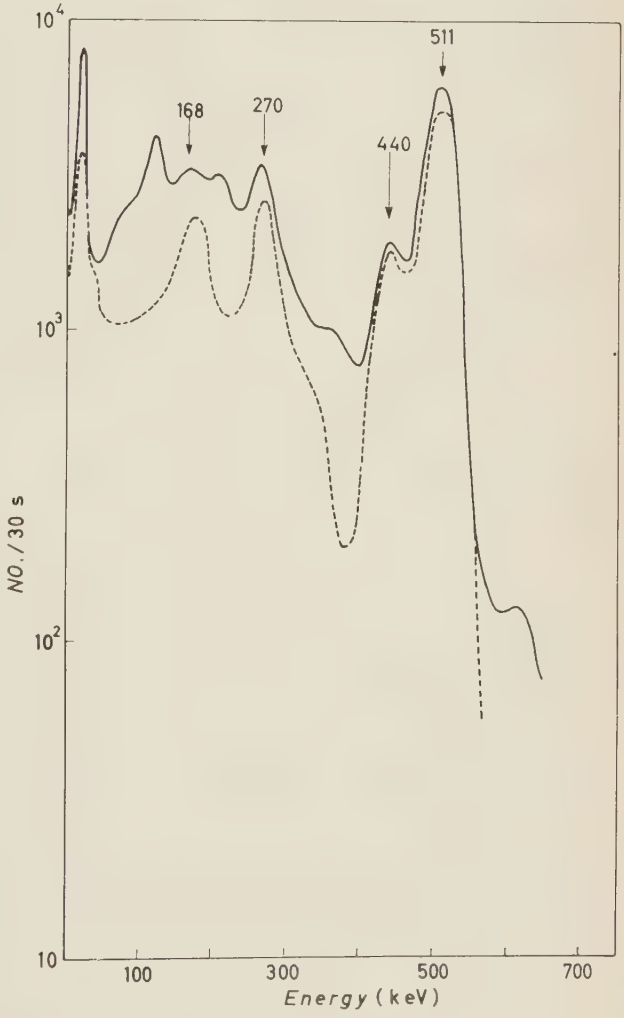

Fig. 2. – Spectrum of low energy γ-rays.

high, and with a single channel analyser. The γ-ray spectrum was also studied with a 4 in. $\times$ 4 in. NaI(Tl) crystal and a 20-channel pulse-height analyser set-up. An expanded spectrum of γ-rays up to 600 keV in energy is shown in Fig. 2, where the full line gives the spectrum as actually taken, and the dotted line shows the spectrum corrected for the background and the contribution of other barium isotope impurities. The peaks due to 168 keV, 270 keV, 440 keV and 511 keV γ-rays were assigned to $^{128}Ba \rightarrow$ $\rightarrow {}^{128}Cs \rightarrow {}^{128}Xe$ from the measurement of the half-life of the decay of their intensities. The spectrum of γ-rays was studied for about a month and the impurities were identified as ^{131}Ba, $^{133}Ba^{m}$.

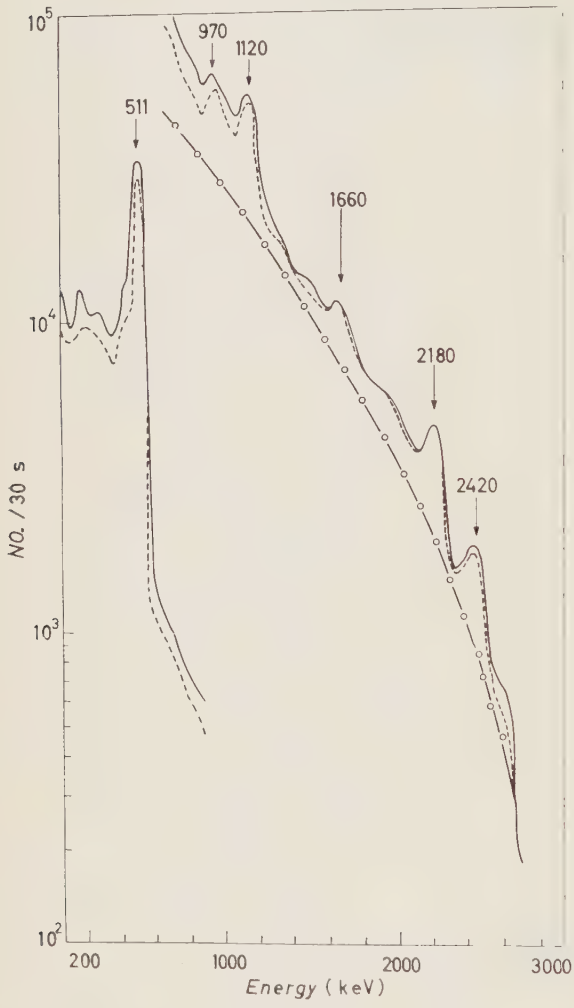

Fig. 3. – Spectrum of high energy γ-rays. The higher energy portion is plotted multiplied by a factor 100.

Fig. 3 gives the higher energy part of the spectrum taken with a single-channel analyser. In order to record weak intensity γ-rays, 1.2 cm of Pb were inserted between the source and the crystal to attenuate the lower energy γ-rays and so avoid the overloading of the spectrometer. The decay of the intensity of the γ-rays in this part of the spectrum also was studied for about a month. The full-line curve in Fig. 3 gives the actual spectrum, the dotted curve gives the spectrum corrected for the background and impurities, and the dashed-curve gives the calculated contribution due to the bremsstrahlung produced by the annihilation of positrons in flight ([10]). For the lead and the

([10]) J. B. GERHART, B. C. CARLSON and R. SHERR: *Phys. Rev.*, **94**, 917 (1954).

NaI(Tl) assembly used in the present experiment, the calculation of the contribution of the bremsstrahlung is based on the experimentally observed ([11]) contribution in the case of pure positron emitters as observed with 1 in. × 1 in. NaI(Tl) crystal. This introduces some uncertainty in the estimate of the intensity of the weak high energy γ-rays. The γ-ray spectrum was analysed with the help of the spectra taken in identical geometry of standard γ-rays, by successively peeling off the contribution of γ-rays starting from the high energy side. As a result of the analysis, taking into account the photo-efficiencies, the energy and the relative intensities of the γ-rays were calculated as given in Table II.

TABLE II.

Energy of the γ-ray in keV	Relative intensity
168±10	10 ±2.0
270±10	23 ±1.5
440±10	50 ±5.0
511	200
970±10	1.5 ±0.3
1 120±15	2.1 ±0.4
1 660±15	0.4 ±0.1
2 180±30	0.36±0.1
2 420±30	0.2 ±0.05

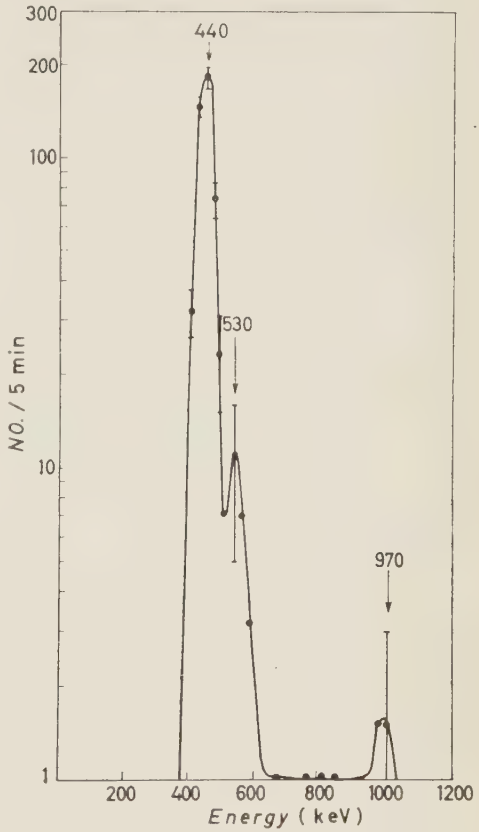

Fig. 4. – Spectrum of γ-rays in coincidence with positrons. The channel-width is 22 keV.

5. – Coincidence studies.

The following coincidence studies of the radiations of ^{128}Cs were made. In coincidence with the positrons, the spectrum of γ-rays below 1 MeV was taken with a coincidence resolving time of $2 \cdot 10^{-7}$ s. The positrons were detected in an anthracene crystal mounted on a Dumont 6292 photomultiplier, the entire positron spectrum above 500 keV was taken in the gate. As can be seen from Fig. 4, 440 keV, 530 keV and 970 keV γ-rays were observed in coincidence with the positrons. From this spectrum, the ratio of the intensities of the stop-over

([11]) R. K. GIRGIS: *Thesis* (Amsterdam, 1960).

530 keV and the cross-over 970 keV γ-ray was found to be about 4. The value for this ratio in the decay of ^{128}I has been found to be 4 ([11]).

The spectrum of γ-rays in coincidence with the annihilation radiation taken with a 20-channel pulse-height analyser is given in Fig. 5. It may be stated that the gate included the 530 keV γ-ray also, which is very much weaker in

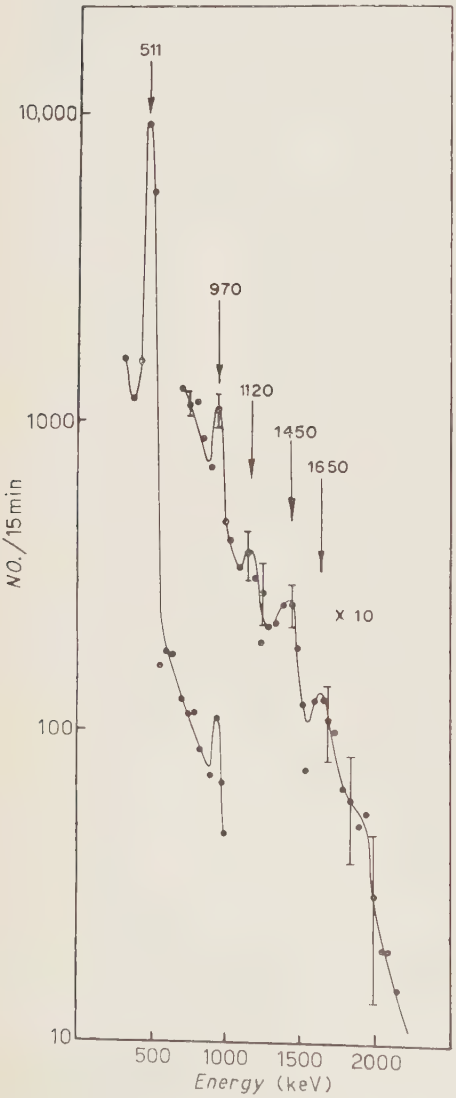

Fig. 5. – Spectrum of γ-rays in coincidence with the annihilation radiation. The gate included the 530 keV γ-ray also. The higher energy portion in the curve is magnified ten times. The channel width is 50 keV.

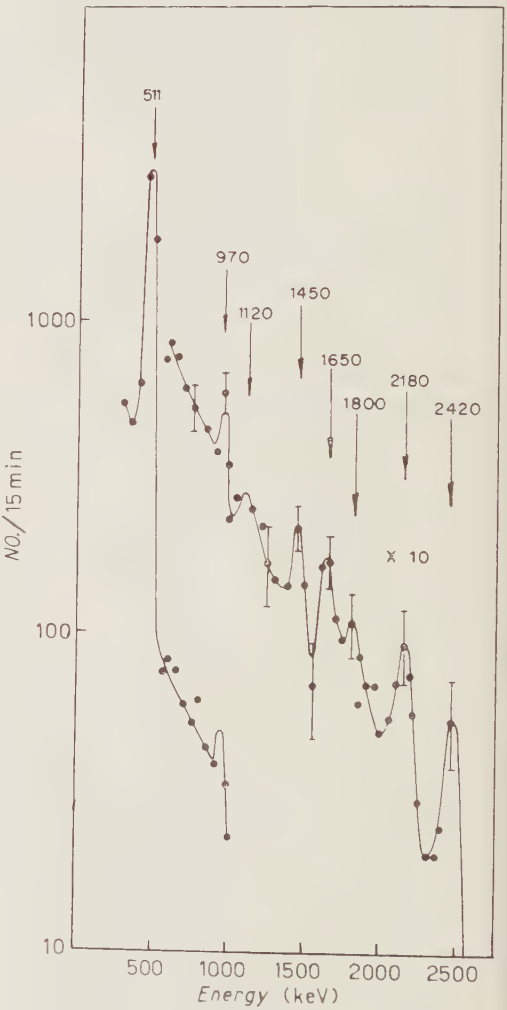

Fig. 6. – Spectrum of γ-rays in coincidence with the 440 keV γ-ray. The gate included some of the annihilation radiation also. The higher energy portion is plotted magnified ten times

intensity. One can see the occurrence of the 511 keV, 970 keV, 1120 keV, 1450 keV, 1650 keV and also probably 1800 keV γ-ray peaks in the coincidence spectrum. The spectrum of γ-rays observed in coincidence with the 440 keV γ-ray taken with the 20-channel pulse-height analyser is reproduced in Fig. 6. The 440 keV gate for this experiment included some annihilation radiation pulses also. In this case, 511 keV, 970 keV, 1120 keV, 1450 keV, 1650 keV, 1800 keV, 2180 keV and 2420 keV γ-rays were observed. From the comparison of the coincidence spectra with 440 and 510 keV gate it was concluded that 2180 and 2420 keV γ-rays are in coincidence only with 440 keV γ-ray while 970 keV γ-ray is in coincidence with either annihilation or 530 keV γ-ray only. The 1120, 1450, 1650 and probably 1800 keV γ-rays seem to be in coincidence with 440 and 511 or 530 keV radiations.

6. – Decay scheme.

With the help of the data on positron spectra and γ-ray spectra and the results of the coincidence experiments, a decay-scheme of ^{128}Cs has been drawn,

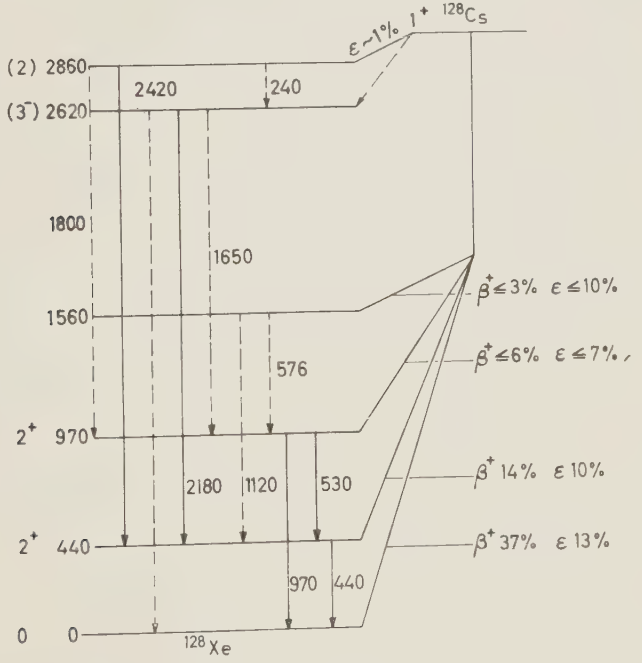

Fig. 7. – A tentative decay scheme of ^{128}Cs.

which is shown in Fig. 7. The levels in ^{128}Xe at 440 keV, 970 keV and 1560 keV can be inferred from the end-point energies of the positron groups

obtained from the Fermi plot of the positron spectrum, the levels at 440 keV and 970 keV being identical with what is observed in the decay of ^{128}I. In addition, two higher levels at 2620 keV, and 2860 keV are shown. These two levels had to be proposed to account for the occurrence of the high energy γ-rays. In the decay-scheme, some transitions are shown with dotted lines because we could not be certain from the results of our measurements that these transitions did take place. This is particularly true of the electron-capture transition to the 2620 keV state.

7. – Discussion.

The log *ft* values for the positron decay from ^{128}Cs (1^+) to the ground, 440 keV, 970 keV and 1560 keV states are all about 5, from which one can conclude that these are all allowed transitions. This is what one would expect for the transitions to the ground (0^-), 440 keV (2^-) and the 970 keV (2^-) states. The spin and parity of the 1560 keV state, consistent with the log *ft* value, could be 0^-, 1^+, 2^-. In the present work, as no γ-ray could be unambiguously taken to originate from the 1560 keV state the assignment to this level of spin and parity 1 seems unlikely. The 1120 keV γ-ray has been shown starting from this state because of the sum-rule consideration and the fact that the 1120 keV γ-ray was found to be in coincidence with the 440 keV γ-ray. We were not able to find any definite evidence of the occurrence of the 576 keV γ-ray, the conversion line of which was detected by NOVAKOV *et al.* ([8]).

On the question whether the 0^+ member of the second vibrational level participates in the decay of ^{128}Cs, we were able to gather no positive evidence, although one would expect this level to figure significantly in the decay scheme of this isotope. Some data have been collected by SAKAI ([9]) showing that the β-decay to the second 2^- state is specially hindered. We have found the data on the β-decay branching to the second 2^- state of the near-spherical nuclei very uncertain. Any conclusion on this question seems unwarranted at the moment. So far as ^{128}I $\rightarrow$ ^{128}Xe is concerned, the log *ft* value for the β-decay to the second 2^- is slightly higher ([12]); our estimate of the log *ft* value for the transition to this state from Cs side shows the same trend.

So far as the 2620 keV state is concerned, it de-excites by transition to 440 keV (2^-) and perhaps also to the ground (0^-) and 970 keV states. Since one cannot rule out, from the results of the present work, the presence of a 240 keV γ-ray arising from the transition from the 2820 keV state to the

([12]) N. BENCZER, B. FARELLY, L. KOERTS and C. S. WU: *Phys. Rev.*, **101**, 1027 (1956).

2 620 keV state, it is not quite certain if the 2 620 keV state is fed directly by the electron-capture decay of ^{128}Cs. If it really turns out that there is no electron-capture branching or negligible electron-capture branching to the 2 620 keV state, a spin larger than two and/or odd parity would have to be assigned to it. One would then be led to assign a spin and parity of 3^- to this state and to suppose that this is the first octupole vibrational state of the even even nucleus ^{128}Xe. The energy of this state and the way it de-excites itself ([13-16]) are tempting enough to incline one to surmise that the 2 620 keV state of ^{128}Xe like the 2 600 keV state of ^{132}Xe ([17]) may be of the octupole vibrational type. Due to the 2 615 keV γ-ray of natural ThC'' background, the relative intensity of the 2 620 keV cross-over transition could not be determined with any accuracy. Roughly speaking, the ratio of the intensities of 2 620 keV and 2 180 keV γ-rays is $4 \cdot 10^{-2}$. The Weisskopf estimate of the ratio on the supposition that the 2 620 keV state is 3^- and that the radiations are pure $E3$ and $E1$, is $4 \cdot 10^{-8}$. It may be pointed out that the intensity of the 2 180 keV γ-ray is about twice that of the 2 420 keV γ-ray. If the 240 keV γ-ray transition alone feeds the 2 620 keV state, its intensity has to be comparable to that of the 2 420 keV γ-ray. It would be hard to reconcile this with normal probabilities of γ-ray transitions.

From the end-point of the positron group arising from ground-to-ground transition, the total decay-energy of ^{128}Cs has been estimated to be (3.907 ± 0.025) MeV. This fact combined with the electron-capture branching to the 2 860 state gives a logft for electron-capture transition to this state equal to 5. The spin and parity of the 2 860 keV state is surmised to be 2^+, consistent with the relative intensities of γ-rays emanating from this state.

In the end, it may be stated that it has not been possible to account for the 1 450 keV γ-ray appearing in coincidence with both the 440 keV γ-ray and the annihilation radiation. It seems plausible to us that there may be other levels near the 1 560 keV level. We have not proposed any, for want of better evidence.

* * *

This work was greatly facilitated by Dr. B. Saraf lending to us the use of his 20-channel pulse-height analyser, for which we owe him a deep debt of gratitude.

([13]) A. M. Lane and E. D. Pendelbury: *Nucl. Phys.*, **15**, 39 (1960).

([14]) B. Cohen: *Phys. Rev.*, **111**, 1568 (1958).

([15]) M. Crut and N. S. Wall: *Phys. Rev. Lett.*, **3**, 520 (1960).

([16]) C. Ythier, W. Schoo, B. L. Schram, H. L. Polak, R. K. Girgis, R. A. Ricci and R. Van Lieshout: *Physica*, **25**, 694 (1959).

([17]) H. G. Devare: to be published.

RIASSUNTO (*)

Presentiamo i risultati di uno studio del decadimento del 2.5 min ^{128}Cs, effettuato con lo spettrometro a raggi β e quello a scintillazione. Il diagramma di Fermi dello spettro dei positroni rilevato con lo spettrometro di Siegbahn-Slätis, dà le seguenti energie nei punti terminali: (2885 ± 15) keV (100, log *ft* 4.8), (2445 ± 25) keV (39, log *ft* 5.1), (1900 ± 40) keV ($\leqslant$ 16, log *ft* $\geqslant$ 5.1) e (1300 ± 40) keV ($\leqslant$ 8, log *ft* $\geqslant$ 4.6). L'analisi degli spettri dei raggi γ rilevati con spettrometri a scintillazione indica i seguenti raggi γ: 168 keV (10), 270 keV (23), 440 keV (50), 511 keV (200), 970 keV (1.5), 1120 keV (2.1), 1660 keV (0.4), 2180 keV (0.36), 2420 keV (0.2). Con l'ausilio di questi dati e di studi sulle coincidenze β^+-γ e γ-γ, si suggeriscono i seguenti livelli del ^{128}Xe: stato fondamentale (0^+), 440 keV (2^+), 970 keV (2^+), 1569 keV (0^+ o 2^+) (tutti alimentati da cattura di positoni ed elettroni), 2620 keV (fase 3^-) e 2860 keV (2^+) alimentati solo da catture di elettroni. Pensiamo che lo stato a 2620 keV possa essere il primo livello vibrazionale ottopolare.

(*) *Traduzione a cura della Redazione.*

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Vapour Pressure of Isotopic Liquids.

III. – Some Corrections to Previous Papers.

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(ricevuto il 3 Dicembre 1960)

Summary. — Some mistakes contained in our previous papers in the derivation of $\ln p_A/p_B$ from the measured α are pointed out. As a consequence, the claimed linear dipendence of $\ln p_A/p_B$ on $1/T$ for argon and neon is to be questioned. The corrected results agree with the measurements by Roth and Bigeleisen, carried out on liquid neon by a direct method.

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1. – As pointed out in the « Note added in proof » of our paper on Ne and A above boiling-point (*) a pretty large discrepancy was noted between our data and those by Roth and Bigeleisen [3]. The main reason for the disagreement is due, as pointed out by Dr. Bigeleisen, to a trivial mistake in sign. The formula at the end of page 513 of II ought to be written

$$\ln\frac{p_A}{p_B} = -\delta_{vl} + \frac{\delta_{vl}^2}{2} - \frac{B(p_A - p_B)}{RT},$$

(*) From now on, our previous papers [1,2] on vapour pressure of isotopic liquids will be denoted by I and II.

([1]) G. Boato, G. Scoles and M. E. Vallauri: *Nuovo Cimento*, **14**, 735 (1959).
([2]) G. Boato, G. Casanova and M. E. Vallauri: *Nuovo Cimento*, **16**, 505 (1960).
([3]) E. G. Roth and J. Bigeleisen: *Journ. Chem. Phys.*, **32**, 612 (1960).

as can be immediately seen from the preceding formula. Moreover we have found some other small errors in the evaluation of the corrections, as explained below.

2. – The approximate equations derived from formula (6) of II are not exactly valid in our experimental conditions. The correct relationship between $\ln \alpha$ and $\ln (p_A/p_B)$ to be applied is found to be, at a very good approximation, the following:

$$\ln \alpha = \ln \frac{p_A}{p_B}\left[1 + \frac{p(B - \tilde{V}_i + C'p)}{RT}\right], \tag{1}$$

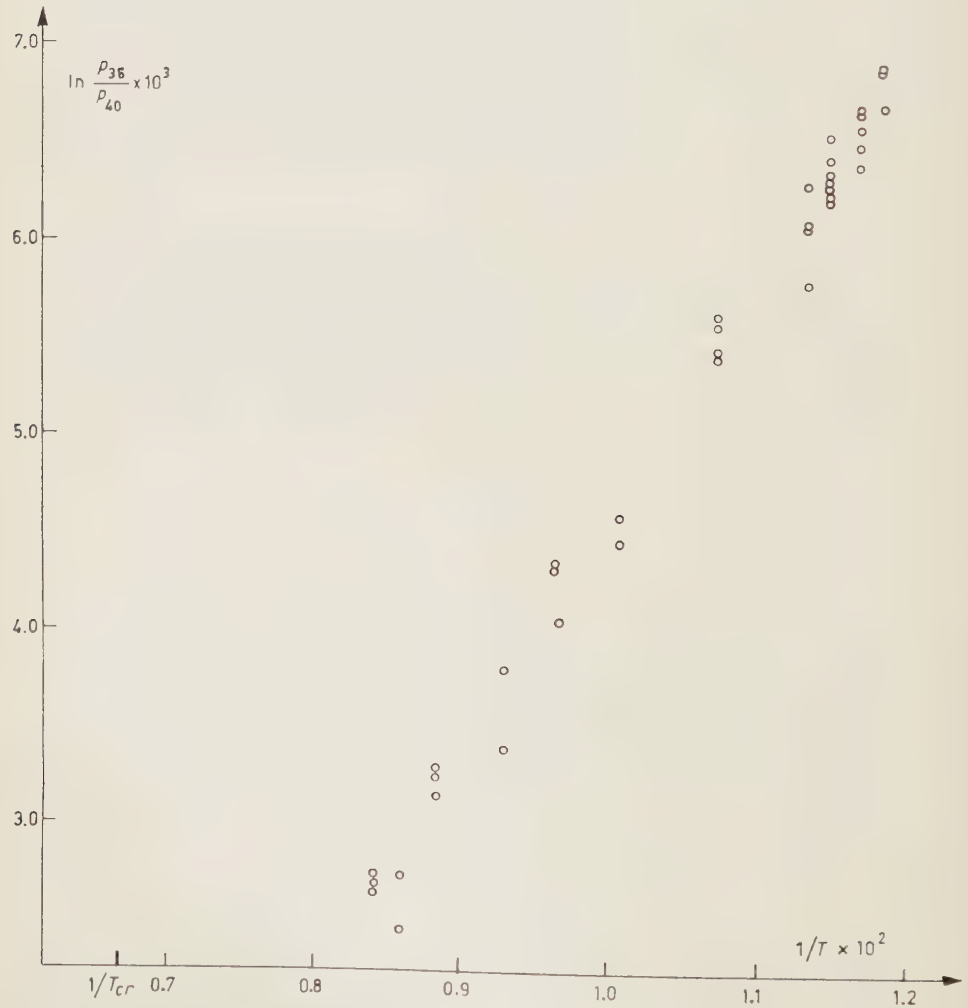

Fig. 1. – Plot of corrected results for argon. In this figure and the following ones the ordinate axis has been shifted in order to get a better resolution of the points.

where $\tilde{V}_l$ is the molar volume of the liquid, p is the total pressure of mixture, and $C' = -B^2/RT$.

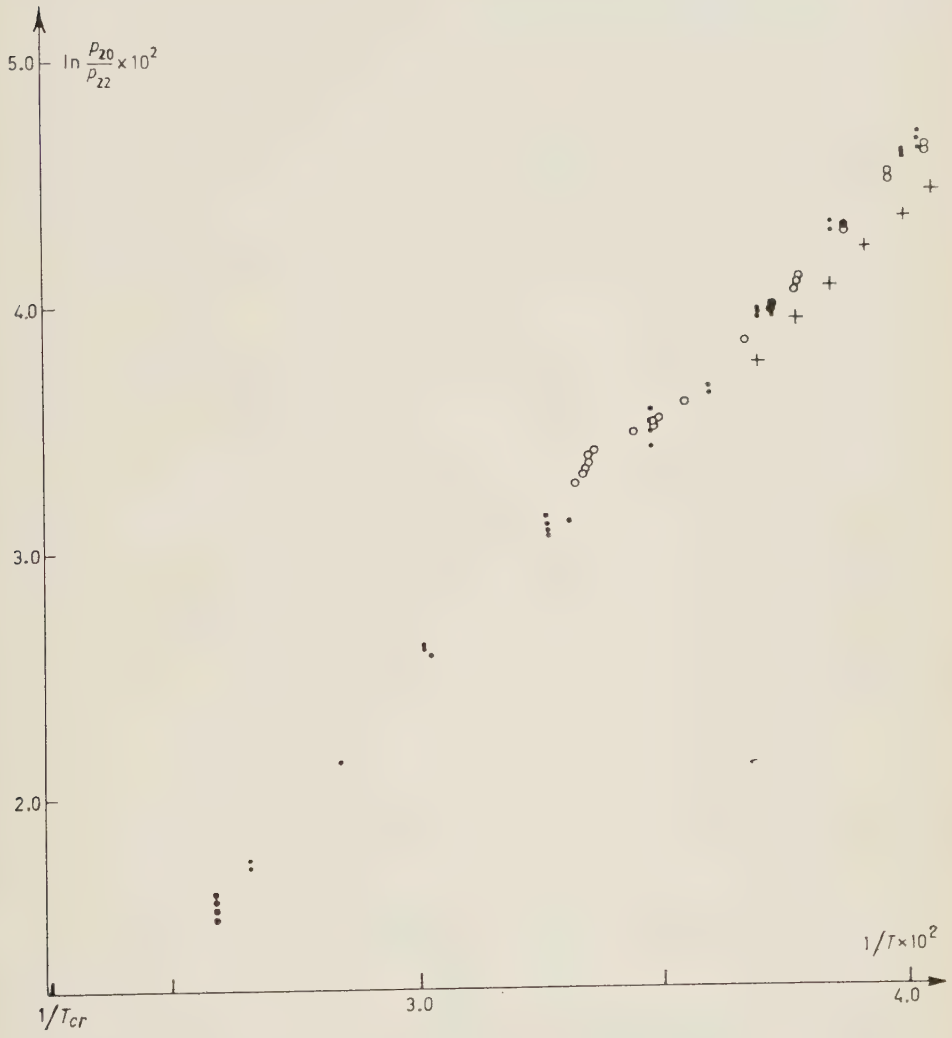

Fig. 2. – Plot of corrected results for neon *vs.* $1/T$ (•). For comparison the results of Keesom and Haantjes (+) and Roth and Bigeleisen (∘) are given.

Owing mainly to the non-ideality of the vapour, the corrective factor within parentheses differs from unity by more than one percent in most cases, even at temperatures below boiling-point. For instance, the corrective factor is 1.02 for argon at triple-point and increases with increasing temperature. Since

our experimental errors are of the order of the percent on the $\ln \alpha$, or better, the identification of $\ln \alpha$ with $\ln (p_A/p_B)$ is never rigorously valid, as assumed in I for A, N_2, O_2 and in II for A.

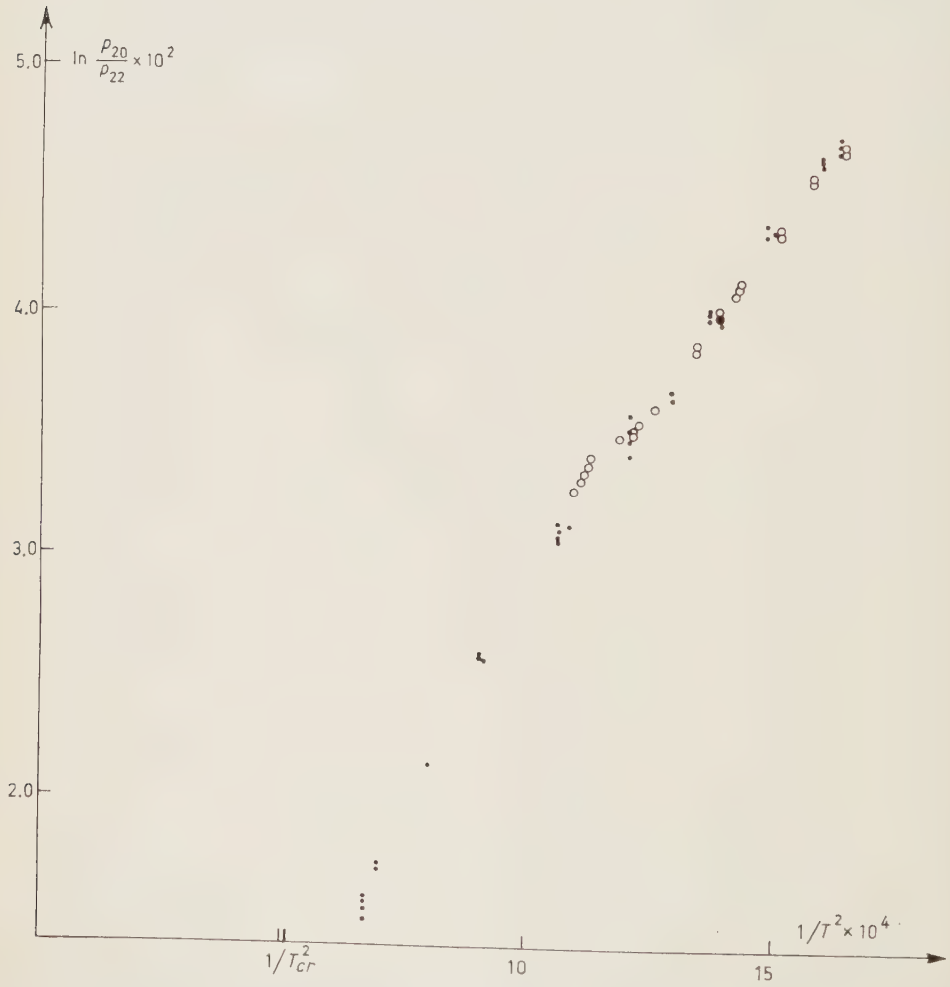

Fig. 3. – Plot of results for neon *vs.* $1/T^2$. ○ Roth and Bigeleisen; ● this paper.

In the case of neon the terms in $\widetilde{V}_l$ and C'' must be taken into account, which was not done in II. When formula (1) is applied, the final results for $\ln (p_A/p_B)$ are somewhat changed, particularly at the highest temperatures. The new results are shown in Tables I and II for argon and neon respectively. Plots of the results *vs.* $1/T$ and $1/T^2$ are shown in Fig. 1, 2 and 3.

TABLE I – *Corrected results for argon.*

Paper I			Paper II		
Sample	T(°K)	$\ln(p_{36}/p_{40})\cdot 10^3$	Sample	T(°K)	$\ln(p_{36}/p_{40})\cdot 10^3$
A 25	84.4	6.8_5	401	93.0	5.6
A 36		6.8_5	402		5.5_5
A 37		6.6_5	403		5.4
			404		5.4
A 21	85.5	6.3_5			
A 23		6.6_5	301	99.2	4.6
A 24		6.4_5	302		4.4_5
A 34		6.6_5			
A 35		6.5_5	201	103.3	4.0_5
A 12	87.0	6.2	405	103.5	4.3_5
A 13		6.3	406		4.3
A 14		6.3			
A 15		6.5	303	107.4	3.8
A 16		6.3	304		3.4
A 32		6.4			
A 33		6.2	407	113.0	3.3
			408		3.2_5
			409		3.1_5
Paper II			305	116.2	2.7_5
Sample	T(°K)	$\ln(p_{36}/p_{40})\cdot 10^3$	306		2.4_5
101	88.2	6.2_5			
102		6.0_5	410	118.7	2.7_5
103		6.0_5	411		2.7
104		5.7_5	412		2.6_5

3. – When the above corrections are made the agreement between our measurements and those by ROTH and BIGELEISEN for liquid neon is completely satisfactory. In fact, the results coincide within the errors of measurements, that is within some permil on the $\ln(p_A/p_B)$. We may now be definitely confident that no systematic error is present in our measurements, at least in the low temperature range.

It appears also that our method of measurement (static determination of α) is completely equivalent to the direct but troublesome comparison of vapour pressure of the pure isotopes (*). In the high temperature range, the correction

(*) Recent measurements on liquid O_2 and N_2 (to be published shortly), show again very good agreement with measurements by the direct method; the discrepancies found in I have practically disappeared.

TABLE II. – *Corrected results for neon.*

Sample	T(°K)	$\ln(p_{20}/p_{22})\cdot 10^2$	Sample	T(°K)	$\ln(p_{20}/p_{22})\cdot 10^2$
101	24.8	4.71	402	28.74	3.52
102		4.68	403		3.48
103		4.65	404		3.42
201	25.02	4.64	303	30.21	3.13
202		4.64			
203		4.61	501	30.64	3.14
			502		3.11
104	25.8	4.33	503		3.08
			504		3.07
204	25.98	4.35			
205		4.31	601	33.06	2.58
105	26.8	3.99	304	33.17	2.61
106		3.99	305		2.60
107		3.96			
			602	35.15	2.14
206	27.01	3.99			
207		3.98	603	37.62	1.74
208		3.96	604		1.74
301	27.79	3.67	505	38.58	1.60
302		3.65	506		1.60
			507		1.55
401	28.74	3.58	508		1.51

to the measured δ's coming from formula (5) of II and from formula (1) of this paper are so large that it is difficult to avoid some unpredictable systematic error.

As a consequence of the mistakes, our tentative conclusions (see II, end of page 515), concerning: 1) the linear dependence of $\ln(p_A/p_B)$ on $1/T$ in the whole range of existence of the liquid, 2) the vanishing of $\ln(p_A/p_B)$ at the critical point, must be dropped out. These findings were due to a fortuitous circumstance and in fact they were difficult to justify theoretically.

The present plots show that a non-vanishing effect is present at the critical point as it is expected from the quantum theorem of corresponding states [4]. As far as the temperature dependence is concerned, no definite conclusion can be easily derived from present data. However Fig. 3 seems to suggest, for

[4] G. BOATO and G. CASANOVA: *Physica*, in print.

neon, a linear dependence of $\ln p_A/p_B$ on $1/T^2$ rather than $1/T$, as pointed out to us by Dr. BIGELEISEN (5). A dependence on $1/T^2$ may be justified by the general theory of small quantum effects (6), but it is not clear how well this theory can be applied to the liquid state.

(5) J. BIGELEISEN: private communication.
(6) K. F. HERZFELD and E. TELLER: *Phys. Rev.*, **54**, 912 (1938).

RIASSUNTO

Un errore di segno e un errore di calcolo nella deduzione di $\ln p_A/p_B$ del fattore di frazionamento α, da noi precedentemente misurato per l'argon e il neon liquidi, pongono in discussione la dipendenza lineare di $\ln p_A/p_B$ da $1/T$ da noi postulata nel precedente lavoro. D'altra parte i risultati così corretti concordano perfettamente con quelli di ROTH e BIGELEISEN, ottenuti per il neon con un metodo di misura completamente diverso.

The Pick-Up Reaction on the Nuclides ^{45}Sc, ^{51}V, ^{52}Cr, ^{54}Fe, ^{56}Fe, ^{55}Mn, ^{59}Co, ^{58}Ni, ^{60}Ni.

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(ricevuto il 6 Dicembre 1960)

Summary. — Energy spectra of deuterons from (n, d) reaction on elements belonging to the $f_{\frac{7}{2}}$ proton shell are presented. Deuteron peaks corresponding to the fundamental and some excited levels of the residual nucleus are brought into evidence. A comparison between relative experimental intensities of ground state transitions, and those predicted on the basis of the shell model hypothesis and the Butler's pick-up mechanism for the n, d reaction is made. The agreement found is rather satisfactory.

1. – Introduction.

The proton configuration of the nuclides ^{45}Sc, ^{51}V, ^{52}Cr, ^{54}Fe, ^{56}Fe, ^{55}Mn, ^{59}Co, ^{58}Ni, ^{60}Ni in the shell model description with j-j coupling can be represented with the symbol $(f_{\frac{7}{2}})^n$, where n is the number of protons which are found in the shell model orbit with angular momentum $l = 3$ and total angular momentum $j = 7/2$. All the other protons are bound in a core formed by the completely filled orbits and ending with the 20-th proton.

In this description, the first of these nuclides, ^{45}Sc, initiates the $f_{\frac{7}{2}}$ orbit with one proton, and the two nickel isotopes have 8 $f_{\frac{7}{2}}$ protons, that is have this orbit completely filled.

If we consider the behaviour of these nuclei in a (n, d) pick-up reaction, we can predict that many similarities will be found.

The (n, d) pick-up reaction is indeed a direct process, where a proton is taken out from the nucleus by an incident neutron to form a deuteron, which is emitted as the final product of the reaction.

The behaviour of this reaction is therefore completely determined by the condition in which the proton is found. Consequently a comparison of the deuteron energy spectra, obtained from these nuclei under neutron bombardment seems to be particularly interesting.

Let us first consider the deuterons emitted in the reaction leading to the ground state of the final nucleus.

These deuterons certainly correspond to having picked-up one of the less bound protons from the initial nucleus, that is in our case an $f_{\frac{7}{2}}$ proton.

Following the description of the pick-up reaction as given by BUTLER [1], in this case we have an angular momentum transfer $l = 3$ due to the fact that we have taken out from the nucleus just a proton with angular momentum $l = 3$.

As the transferred angular momentum determines the angular distribution of the reaction products, we expect for the deuterons belonging to the maximum energy peak an angular distribution very similar in all the nuclei, as it can be calculated by means of Butler's theory.

This must hold for all the nuclei here discussed. It is to be stressed that the preceeding discussion is valid if the two following points are valid:

1) the nuclei considered can be exactly described by means of the shell model with j-j coupling,

2) the reaction mechanism is the pick-up as described by Butler's theory.

A study of (n, d) reactions leading to the ground state of the final nucleus on the elements discussed above allows us to test the validity of these two assumptions.

The study of the deuteron energy distribution is very interesting, with the aim to see the contribution to the reaction coming from the different excited states of the residual nucleus.

[1] S. T. BUTLER and O. H. HITTMAIR: *Nuclear Stripping Reactions* (New York, 1957).

2. – Experimental apparatus.

The technique used for deuteron detection is the same described in some previous work ([2,3]). All the measurements described in the following have been taken at an angle for deuteron emission of about 14°, in respect to the incident neutron beam. The angular aperture was of about 28°, the neutron energy 14 MeV.

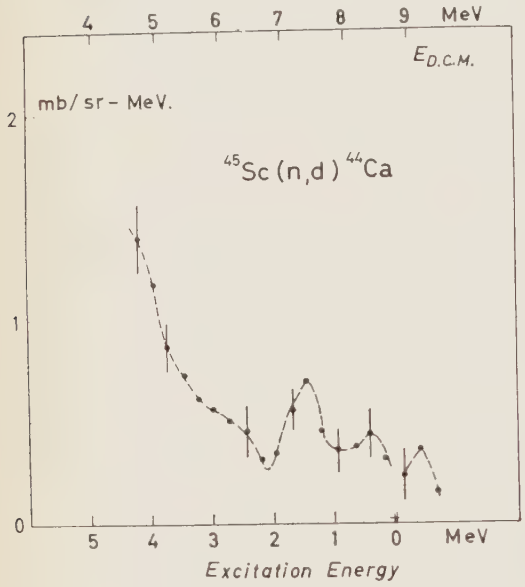

Fig. 1. – Deuteron energy spectrum from ^{45}Sc. The deuteron energy (upper scale) and residual nucleus excitation energy (lower scale) are shown on the abscissae. On the ordinate is shown the cross-section in mb/sterad MeV.

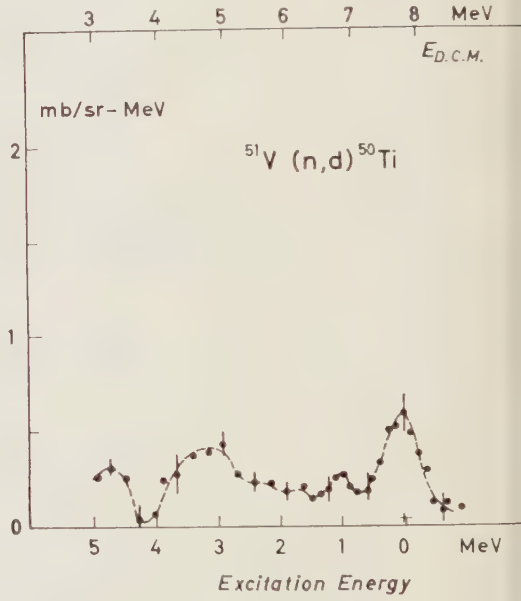

Fig. 2. – Deuteron energy spectrum from ^{51}V.

The energy resolution is of about 5% at 10 MeV and 30% at 3 MeV mainly due to the thickness of the target. The calibration of the deuteron energy scale and of the absolute value of the cross-section is made by means of recoil deuterons obtained from a thin layer of deuterated paraffin.

The deuteron energy spectra obtained in our measurements are shown in the figures from 1 to 9.

([2]) G. MARCAZZAN, A. M. SONA and M. PIGNANELLI: *Nuovo Cimento*, **10**, 155 (1958).
([3]) L. COLLI, F. CVELBAR, S. MICHELETTI and M. PIGNANELLI: *Nuovo Cimento*, **14**, 1220 (1959).

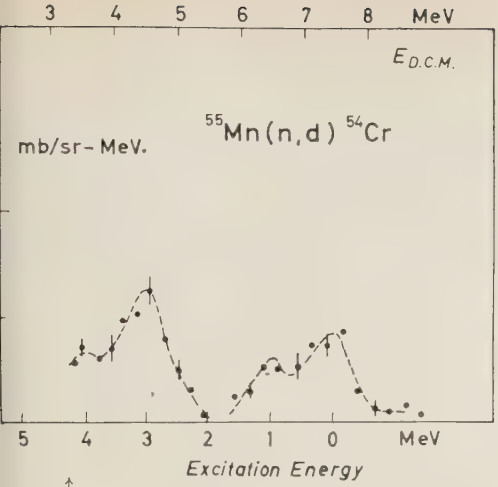

Fig. 3. – Deuteron energy spectrum from ^{52}Cr.

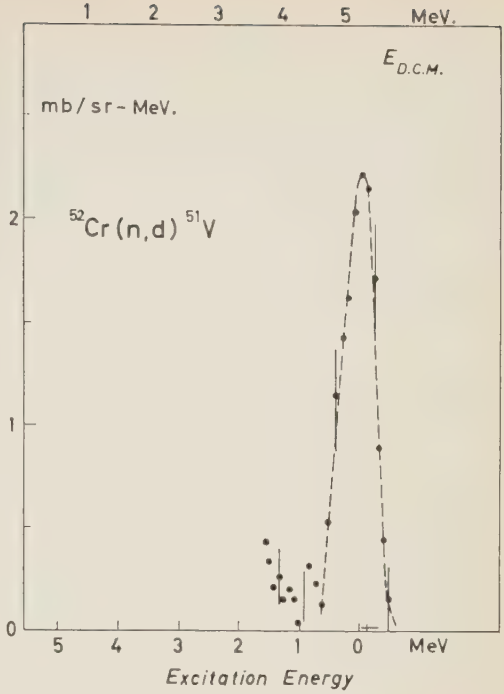

Fig. 4. – Deuteron energy spectrum from ^{55}Mn.

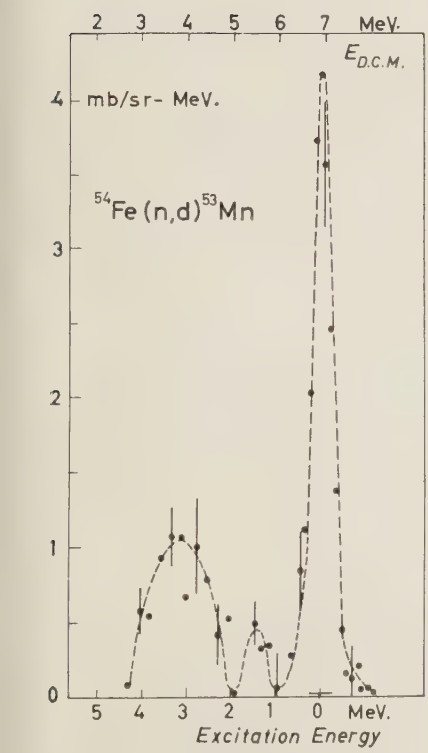

Fig. 5. – Deuteron energy spectrum from ^{54}Fe.

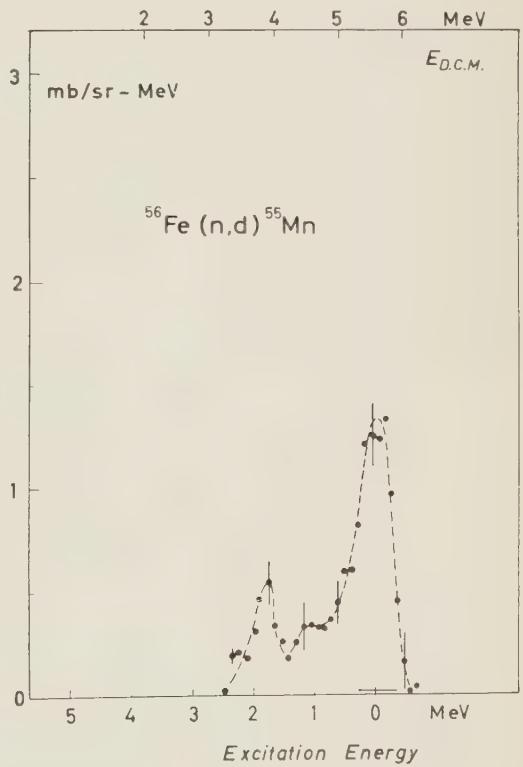

Fig. 6. – Deuteron energy spectrum from ^{56}Fe.

On the horizontal axis, two energy scales are shown: the kinetic energy of emitted deuterons in c.m., and the excitation energy of the residual nucleus.

The vertical axis gives the absolute value of the cross-section. The spectrum of the $^{55}Mn(n, d)^{54}Cr$ reaction has been included, which has been already published in a previous work [3], because the published spectrum was found to be wrong due to an error in the background subtraction.

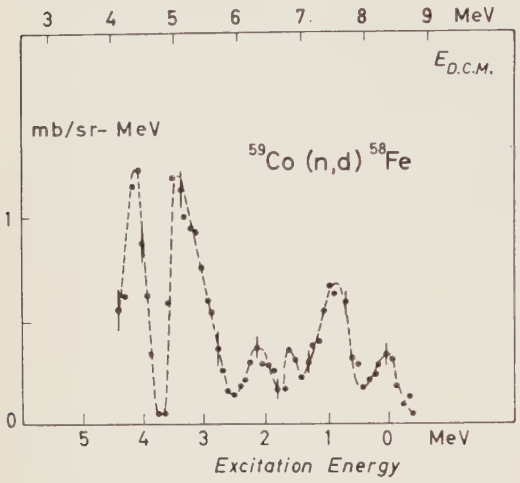

Fig. 7. – Deuteron energy spectrum from ^{59}Co.

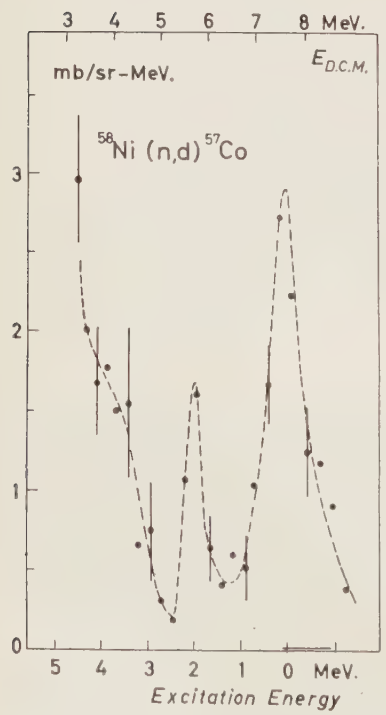

Fig. 8. – Deuteron energy spectrum from ^{58}Ni.

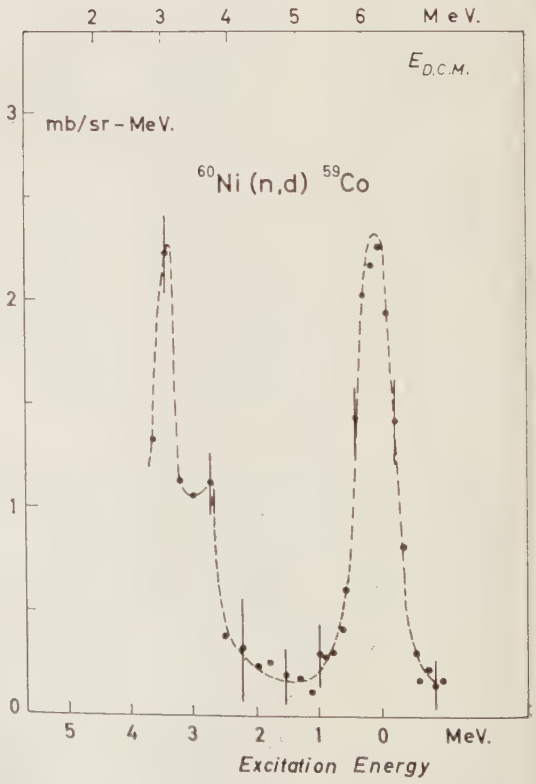

Fig. 9. – Deuteron energy spectrum from ^{60}Ni.

When the mass data ([4]) are known with small errors, we found a good agreement between the calculated maximum deuteron energy and our results. This fact shows that our energy scale is rather precise, and allows us to give a more precise Q value for other cases and correct it to a higher value for two cases (^{55}Mn and ^{58}Ni).

3. – Discussion.

The results here presented give evidence of a number of characteristic features of the (n, d) reaction.

As to the spectrum shape, it can be noted that:

1) in many cases, the contribution to the reaction of maximum energy deuterons is predominant, that is the residual nucleus has a relatively high probability to be left in the fundamental state,

2) the residual nucleus excited states appear in the spectra with different intensities, which shows that the reaction mechanism makes a selection on the states of the residual nucleus.

It is particular interesting a peak at ~ 3 MeV of excitation energy in 5 of the studied spectra (^{59}Co, ^{55}Mn, ^{51}V, ^{54}Fe, ^{60}Ni).

We can, at first sight, divide the nine spectra in two classes, depending on the proton number. The spectra obtained from an initial nucleus with an even proton number (Cr, Fe, Ni) have generally a rather big cross-section for the maximum energy deuteron peak, while those obtained from nuclei with an odd proton number (Sc, V, Mn, Co) have the corresponding cross-section much smaller.

We want to analyze this point in more detail. The quantity related to the nuclear structure that can be deduced from the pick-up cross-section is the « reduced width » belonging to the transition considered.

The « reduced width » can be easily calculated using the formula given by Macfarlane and French ([5])

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} = 183.6 \frac{A^2}{(A+2)^2} \left(\frac{E_\mathrm{d}}{E_\mathrm{n}}\right)^{\frac{1}{2}} \cdot \sigma^l_{\mathrm{Tab}} \cdot r_0^3 \cdot \theta^2 , \tag{1}$$

where $\mathrm{d}\sigma/\mathrm{d}\omega$ is the differential cross-section at a fixed angle, A the mass of the residual nucleus, E_d and E_n the kinetic energy of the deuteron and of the

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([4]) A. H. Wapstra: *Physica*, **71**, 367 (1955); H. E. Duckworth: *Prog. Nucl. Phys.*, **6**, 155 (1957).

([5]) M. H. Macfarlane and J. B. French: *Rev. Mod. Phys.*, **38**, 567 (1960).

neutron involved, σ^{l}_{Tab} a function containing the entire angular dependence, tabulated by LUBITZ ([6]), r_0 the interaction radius and θ^2 the reduced width.

In the case, of deuteron peak corresponding to ground state transition as it was discussed above, the expected transferred angular momentum is $l = 3$. A recent measurement made on the maximum energy deuteron peak of ^{51}V(n, d)^{50}Ti by CINDRO *et al.* ([7]) confirms this point.

Therefore, we know that at our angle of measurement, the value of $\sigma(l = 3)$ is very small, about 1/10 or less of its value at maximum, which should be found at about 40°.

Moreover, any deviation from the strictly considered Butler's theory, like the effect of wave distortion for the incident and outgoing particle due to the interaction with the nuclear potential can change the value of σ^{l}_{Tab} at this angle by an appreciable quantity.

Therefore, as we have not yet measured the deuteron angular distribution, in extracting the reduced widths from the cross-sections we must make the hypothesis that deuterons have strictly an angular distribution as given by Butler's theory. The only correction that we are able to do is the one due to the Coulomb interaction, which is easily taken into account, in the case of a (n, d) reaction.

This fact does not encourage us to give much meaning to the values of the reduced width calculated from the formula (1).

Due to that, we will leave any consideration about the absolute value of the reduced width to the moment when the deuteron angular distribution will be known, and will restrict ourselves to a relative comparison of the reduced widths belonging to the ground state transitions for the different nuclei.

To understand the meaning of the reduced width, we will follow MACFARLANE and FRENCH, and, using their notation, divide the « reduced width » θ^2 in two factors:

$$\theta^2 = S\theta_0^2 ,$$

where θ_0^2 is the « single particle reduced width » and S is called the « spectroscopic factor ». θ_0^2 gives the probability for a nucleon in a potential well to come out, and S is the probability that the initial state be similar to the final state, that is with one particle separated by the others in a single particle state.

On the basis of the shell model with jj coupling scheme, the spectroscopic factor S has been calculated, with the introduction of the coefficients of fractional parentage.

([6]) C. R. LUBITZ: *Numerical Table of Butler-Born Approximation*, Randall Laboratory of Michigan University (1957).

([7]) N. C. CINDO, I. ŠLAUS and P. THOMAS: private communication and work presented to the *Annual Conference of the S.I.F.* (Naples, Sept. 1960).

This is greatly simplified if we can ignore the neutron group, calculating the reduced width for n protons in the $f_{\frac{7}{2}}$ shell. This simplification can certainly be done for all the considered nuclei (except ^{45}Sc), because all of them have a neutron configuration with the $f_{\frac{7}{2}}$ orbit completely filled, and, in this situation the neutrons do not affect proton emission.

For ^{45}Sc this argument is not valid, because the $f_{\frac{7}{2}}$ orbit for neutrons is not filled, but we will disregard neutrons all the same, in view of the great simplification.

In the hypothesis discussed above, the spectroscopic factor S corresponding to the transition $n \to n-1$, where n is the number of protons in the last shell in the initial nucleus is given by the simple formulas:

$$\text{(II)} \qquad \begin{aligned} S(n \to n-1) &= n && \text{if } n \text{ is even}, \\ S(n \to n-1) &= 1 - \frac{n-1}{2j+1} && \text{if } n \text{ is odd}. \end{aligned}$$

In our cases, for n varying from 1 to 8 we have

$$S = 1,\ 2,\ \tfrac{3}{4},\ 4,\ \tfrac{1}{2},\ 6,\ \tfrac{1}{4},\ 8\,.$$

Reduced widths calculated from the formula (I) can be compared with these numbers in relative values.

Before doing so, we must discuss still one parameter of the formula (I) that is the interaction radius r_0.

The measurements of angular distribution of deuterons from ^{51}V(n, d) show an interaction radius in agreement with the formula

$$\text{(III)} \qquad r = (1.22\,A^{\frac{1}{3}} + 1.7)\cdot 10^{-13}\ \text{cm}\,.$$

Some stripping experiments on the nuclei ^{51}V and ^{52}Cr and ^{48}Ti show a radius also in agreement with this formula ([8]). This is not true for ^{59}Co, for which a much smaller radius is found ([9]).

In a very recent experiment on (p, p′) scattering with excitation of collective states of the residual nucleus, an interaction radius is found which is somewhat smaller for nuclei near the closed shells than for the one far from them ([10]).

([8]) A. W. DALTON, A. KIRK. G. PARRY and H. D. SCOTT: *Proc. Phys. Soc.*, **75**, 95 (1960); EL BEDEWI-TADROS: *Nucl. Phys.*, **8**, 71 (1958); A. J. ELWYN and F. B. SHRILL: *Phys. Rev.*, **111**, 925 (1958).

([9]) EL BEDEWI-TADROS: *Nucl. Phys.*, **8**, 79 (1958).

([10]) H. FARAGGI: private communication and work presented to the *Annual Conference of the S.I.F.* (Naples, Sept. 1960).

The last consideration seem to suggest the use of smaller interaction radius for the nuclei Co and Ni, that are indeed very near to closed shell (Co) or closed shell nuclei (Ni).

The results of the calculation of the reduced width is shown in Table I. Here the following quantity are shown, all referring to the maximum energy deuteron peak: Q-value, cross-section value, r_0 value as given by the formula (III) theoretical value of the spectroscopic factor S and experimental value of S relative to the ^{54}Fe value, which is put equal to 6.

TABLE I.

Element	Q(MeV)	$\sigma(14°)$ mb/st	r_0 (fermi)	$S_{th.or}$	S_{exp}
^{45}Sc	—	$\leqslant 0.30$	6.01	1	0.29
^{51}V	-5.88 ± 0.15	$0.48\pm10\%$	6.19	0.75	0.72
^{52}Cr	-8.42 ± 0.15	$1.11\pm10\%$	6.25	4	3.85
^{55}Mn	-6.31 ± 0.20	$0.35\pm20\%$	6.31	0.5	0.59
^{54}Fe	-6.65 ± 0.15	$2.44\pm10\%$	6.29	6	6
^{56}Fe	-7.93 ± 0.15	$1.15\pm10\%$	6.34	6	3.25
^{59}Co	—	$0.15\pm30\%$	6.42(5.65)	0.25	0.15(0.25)
^{58}Ni	-5.95 ± 0.15	$2.84\pm10\%$	6.40(5.65)	8	3.59(8)
^{60}Ni	-7.40 ± 0.15	$1.90\pm10\%$	6.45(5.65)	8	5.15(9.15)

The agreement between theoretical and experimental values is rather good, for the nuclei ^{51}V, ^{52}Cr, ^{55}Mn, ^{54}Fe; to bring in agreement the experimental and the theoretical value of S for ^{59}Co and the two Ni we could choose a smaller radius, following what said above: indeed with a radius $r_0 = 5.65$ these three nuclei give respectively the value $S = 0.25$, 8, 9.15. The other two nuclei ^{45}Sc and ^{56}Fe have both too small experimental S value, relative to ^{54}Fe. In the case of Sc, the explanation can probably be found in the fact that we have not used isotopic spin formalism. For the ^{56}Fe it must be noted that the final nucleus of this reaction is ^{55}Mn, that is a nucleus with spin $\frac{5}{2}$ in the ground state.

This anomalous spin value shows that this nucleus is not correctly interpreted on the basis of shell model with jj coupling, because this predicts 7/2.

Due to this anomaly, the transition in question cannot be considered like a transition between single particle shell model states, and this certainly reduces the transition probability.

The same nucleus is involved in the reaction ^{55}Mn(n, d)^{54}Cr, now being the initial nucleus. This case must therefore be considered with care, even if it seems to show agreement.

In conclusion, the relative reduced widths as deduced by our results seem to show that at least a qualitative agreement exists between the prediction

of the shell model and the (n, d) results. To go deeper in the analysis, the deuteron angular distribution is needed.

* * *

We gladly thank Prof. A. AGODI for the very interesting discussions on this subject.

RIASSUNTO

Sono presenti gli spettri di energia dei deutoni emessi dagli elementi i cui ultimi protoni appartengono alla shell $f_{\frac{7}{2}}$. Vengono messi in evidenza picchi di deutoni corrispondenti alle transizioni che conducono sia allo stato fondamentale del nucleo residuo che ad alcuni stati eccitati. Viene fatto un confronto tra le intensità delle transizioni allo stato fondamentale trovate sperimentalmente e quelle calcolate in base alle ipotesi che la reazione avvenga per mezzo di un meccanismo di pick-up e che i nuclei interessati siano descrivibili in base al modello a shell. L'accordo è abbastanza soddisfacente.

On Plasma Oscillations with Special Emphasis on the Landau Damping and the Gross Gaps in the Frequency Spectrum.

A. Kildal

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(ricevuto il 17 Dicembre 1960)

Summary. — A linear theory of oscillations of the form $\exp[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)$ in a collision-free electron plasma in an external magnetic field $\boldsymbol{H}_0$ is given. The theory is based on the equations of motion for each particle. Only the case $\boldsymbol{k}\to 0$ is studied and the singularities which then appear, are treated with the use of δ-functions. Expressions for the perturbation $\delta\boldsymbol{v}$ of the electron velocity are found. By a Taylor expansion in $\delta\boldsymbol{v}$ the perturbation of the distribution function is evaluated and the density perturbation N_1 as well. The last quantity also satisfies the divergence equation. On elimination, a dispersion equation is obtained. In the case $\boldsymbol{k}_{\parallel}\boldsymbol{H}_0$, electrons with velocity equal to the phase velocity experience a constant electric field. Being thus accelerated, they deprive the field of its energy. This resonance absorption is calculated and the accompanying damping agrees with the Landau damping. Next the frequency gaps which appear in the case $\boldsymbol{k}\cdot\boldsymbol{H}_0 = 0$ are studied.

1. – Introduction.

In recent years there has appeared a lot of papers treating oscillations in an unbounded, quasi-neutral electron-plasma with, or without external fields. In some of the papers the hydromagnetic approximation is taken into use, which together with the equations of Maxwell constitutes a full set of equations for the determination of quantities of interest. This method, however has its flaws as compared to another method of description, namely the use of Boltzmann's kinetic equation together with the equations of Maxwell.

Especially, when short-range collisions are neglected, the last method predicts effects which the former fails to do. Examples are the Landau damping and the Gross gaps, the last effect appearing only in the presence of an external magnetic field H_0. Later in this paper we shall point out the reason why these effects do not appear in the hydrodynamic description.

When studying pure longitudinal electron oscillations in a collision-free, unmagnetized plasma, one arrives at a dispersion equation in the form of a singular integral

$$1 = \frac{4\pi e^2 N_0}{mk^2} \int_{-\infty}^{\infty} \frac{f_0'(v)}{v - \omega/k} \, \mathrm{d}v \,. \tag{I}$$

Here m, $-e$, and N_0 are the mass, charge and equilibrium density of electrons, k, ω = wave number and wave frequency, $f_0(v)$ – distribution function for the electrons in equilibrium.

We see from (I) that the integral is not defined since the integrand turns to infinity when $v \to \omega/k$ and $f_0'(\omega/k) \neq 0$. The physical and mathematical reason for the singularity as well as the treatment of the integral have been much discussed by different authors.

VLASOV [1] which first derived the relation (I) as a result of assuming wave solutions of the form $\exp[i(kx - \omega t)]$ in Boltzmann's equation, decreed that the principal value should be taken. He did not, however, give any reason for this procedure.

LANDAU [2] claimed that the substitution method was not valid for the problem in question, and subjected instead the Boltzmann's equation to a Laplace transform. LANDAU arrived at the same dispersion equation as that found by VLASOV except for a change in the integration path, a consequence of which was a damping of the waves. This damping is small for small wave numbers k, but increases with increasing k.

BOHM and GROSS [3] in their papers are of the opinion that the linear treatment can't be used when electrons with velocity ω/k are present, and put $f_0(v) = 0$ in an interval around $v = \omega/k$.

VAN KAMPEN [4] finds this procedure unsatisfactory from a mathematical point of view and asserts that the singularity is of mathematical origin and will disappear when treated properly. He further disagrees with LANDAU on the validity of the substitution method, and shows that this method leads to the same result as found previously by LANDAU.

The above mentioned papers are only a few of the many publications which have appeared on plasma oscillations. The subject has also been treated in some recently published books. As concerns the physical reason, there is still some discussion and disagreement. There seem to be three different opin-

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(1) A. A. VLASOV: *Žurn. Èksp. Teor. Fiz.*, **9**, 25 (1945).
(2) L. LANDAU: *Žurn. Èksp. Teor. Fiz.*, **10**, 25 (1946).
(3) D. BOHM and E. P. GROSS: *Phys. Rev.*, **75**, 1851 (1949).
(4) N. G. VAN KAMPEN: *Physica*, **21**, 949 (1955).

ions of the physical cause of Landau damping, stating that it is due

1) to trapped electrons,
2) to phase mixing (diffusion damping),
3) to a resonance phenomenon.

In the present paper, the theory of plasma oscillations will be treated by help of a « particle description », thus using Newton's second law for each particle. We further assume solutions of the form $\exp[i(kx - \omega t)]$, where x is the direction of propagation. We will show that our results are in conformity with the results of LANDAU and VAN KAMPEN. From a mathematical point of view, the present method is rather simple and unveils the physical reason for the Landau damping as being due to just those electrons having a velocity equal to the phase velocity of the wave. In the present author's opinion, the cause of the damping is a resonance phenomenon, and the third cathegory is thus supported.

The same mathematical method is also used to describe another phenomenon, namely gaps in the frequency spectrum first found by GROSS ([5]). These gaps are probably also due to a kind of resonance phenomenon, and are together with the Landau damping an effect which can be fully explained only by a non-linear theory.

BERNSTEIN ([6]) extended the theory of Gross to the case of arbitrary propagation direction and arrived at the same results as GROSS when going to the limit of wave propagation transverse to the external magnetic field H_0, which is the case treated by GROSS.

As there recently (OSTER, ([7])) have been risen certain objections against the results of Gross, we shall give some comments on this effect as well.

2. – The equations of motion and their solution.

The physical conditions are: an unbounded quasi-neutral plasma, consisting of single-charged electrons and ions, embedded in an external, homogeneous and constant magnetic field H_0. As usual, because of their much greater mass, the ions will be considered motionless.

We assume that there has come into existence a small electric field, the cause of which is a perturbation of the neutrality. The equation of motion for the electron is

$$m\ddot{\boldsymbol{r}} = -e\boldsymbol{E} - \frac{e}{c}\dot{\boldsymbol{r}} \times \boldsymbol{H}_0 , \tag{1}$$

([5]) E. P. GROSS: *Phys. Rev.*, **82**, 232 (1951).
([6]) I. B. BERNSTEIN: *Phys. Rev.*, **109**, 10 (1958).
([7]) L. OSTER: *Rev. Mod. Phys.*, **32**, 141 (1960).

or in component form

$$\ddot{x} = -\frac{e}{m} E_x - \omega_c \dot{y} \tag{2a}$$

$$\ddot{y} = -\frac{e}{m} E_y + \omega_c \dot{x} , \tag{2b}$$

$$\ddot{z} = -\frac{e}{m} E_z . \tag{2c}$$

Here $-e$ = electron charge, m = electron mass, $\omega_c = eH_0/mc$ is the cyclotron frequency, $\boldsymbol{E}$ the electric field, and $\boldsymbol{r}$ the position-vector. The direction of $\boldsymbol{H}_0$ is along the z-axis. Let us assume a wave motion of the form

$$\boldsymbol{E} = \boldsymbol{E}_0 \exp\left[i(k_1 x + k_2 z - \omega t)\right] , \tag{3}$$

where the wave number is $k = \sqrt{k_1^2 + k_2^2}$. The direction of propagation may be given by

$$\theta = \operatorname{arctg} \frac{k_1}{k_2} . \tag{4}$$

We introduce Lagrangian co-ordinates by writing

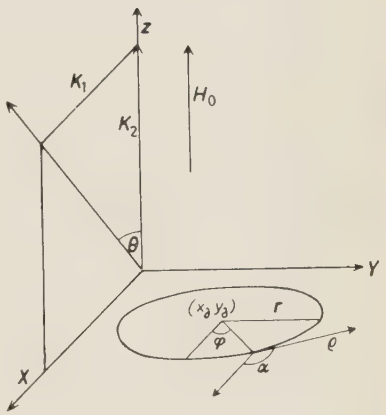

Fig. 1.

$$x = x_0 + r \cos\varphi + \chi , \tag{5a}$$

$$y = y_0 + r \sin\varphi + \eta . \tag{5b}$$

$$z = z_0 + v_{0z} t \quad + \zeta . \tag{5c}$$

Here χ, η, ζ, are disturbances due to $\boldsymbol{E}$, v_{0z} is the unperturbed velocity of the electron in the z-direction, t = time, $\varphi = \varphi_0 + \omega_c t$, x_0, y_0, z_0, φ_0 are initial quantities and r = gyration radius in equilibrium.

We introduce (5) into (2) and get the following equations:

$$D(D\chi) + \omega_c(D\eta) = -\frac{e}{m} E_x \tag{6a}$$

$$-\omega_c D\chi + D(D\eta) = -\frac{e}{m} E_y , \qquad D = \frac{\mathrm{d}}{\mathrm{d}t} \tag{6b}$$

$$D(D\zeta) = -\frac{e}{m} E_z . \tag{6c}$$

The two first eq. (6*a*) and (6*b*) are independent of the third (6*c*), and give when using Cramers' rule symbolically

$$(7)\qquad D\chi = -\frac{e}{m}\frac{\begin{vmatrix} E_x & \omega_c \\ E_y & D \end{vmatrix}}{\begin{vmatrix} D & \omega_c \\ -\omega_c & D \end{vmatrix}}, \qquad D\eta = -\frac{e}{m}\frac{\begin{vmatrix} D & E_x \\ -\omega_c & E_y \end{vmatrix}}{\begin{vmatrix} D & \omega_c \\ -\omega_c & D \end{vmatrix}},$$

or

$$(8)\qquad (D^2+\omega_c^2)D\chi = -\frac{e}{m}(DE_x - \omega_c E_y),$$

$$(9)\qquad (D^2+\omega_c^2)D\eta = -\frac{e}{m}(DE_y + \omega_c E_x).$$

We now make the assumption that the quantities χ, η, ζ are small compared to the wave-length:

$$\chi, \eta, \zeta \ll \lambda = \frac{2\pi}{\sqrt{k_1^2+k_2^2}},$$

which has the effect of linearization. When taking this assumption into account, and using eq. (5) in eq. (3), we get in a first-order approximation

$$(10)\qquad \boldsymbol{E} = \boldsymbol{E}\exp\left[i\{(k_1x_0+k_2z_0)+k_1r\cos\varphi+(k_2v_{z0}-\omega)t\}\right].$$

To enable ourselves to integrate the differential eq. (6*c*), (8) and (9) when the expression (10) for $\boldsymbol{E}$ is inserted, we introduce the series expansion

$$(11)\qquad \exp\left[ik_1r\cos\varphi\right] = \sum_{-\infty}^{\infty} i^n J_n(k_1r)\exp\left[in\varphi\right],$$

where $J_n(k_1r)$ are Bessel functions of the first kind and of the n-th order.

The differential equations to be solved then become

$$(12)\qquad (D^2+\omega_c^2)D\chi = -\frac{e}{m}\exp\left[i\boldsymbol{k}\cdot\boldsymbol{r}_0\right]\sum_{-\infty}^{\infty} i^n\exp\left[in\varphi_0\right]J_n(k_1r)\cdot$$
$$\cdot\left[E_{0x}i(n\omega_c+k_2v_{0z}-\omega) - E_{0y}\omega_c\right]\exp\left[i(n\omega_c+k_2v_{0z}-\omega)t\right],$$

$$(13)\qquad (D^2+\omega_c^2)D\eta = -\frac{e}{m}\exp\left[i\boldsymbol{k}\boldsymbol{r}_0\right]\sum_{-\infty}^{\infty} i^n\exp\left[in\varphi_0\right]J_n(k_1r)\cdot$$
$$\cdot\left[E_{0x}\omega_c + E_{0y}i(n\omega_c+k_2v_{0z}-\omega)\right]\exp\left[i(n\omega_c+k_2v_{0z}-\omega)t\right],$$

$$(14)\qquad D(D\zeta) = -\frac{e}{m}\exp[i\boldsymbol{k}\cdot\boldsymbol{r}_0]\sum_{-\infty}^{\infty} i^n J_n(k_1 r)\exp[i\eta\varphi_0]E_{0z}\cdot$$

$$\cdot\exp[i(n\omega_c + k_2 v_{0z} - \omega)t]\,, \qquad \boldsymbol{k}\cdot\boldsymbol{r}_0 = k_1 x_0 + k_2 z_0\,.$$

For the sake of convenience, we write the eq. (12)–(14) as follows:

$$(15a)\qquad (D^2 + \omega_c^2)D\chi = \sum_{-\infty}^{\infty} a_n \exp[i(n\omega_c + k_2 v_{0z} - \omega)t]\,,$$

$$(15b)\qquad (D^2 + \omega_c^2)D\eta = \sum_{-\infty}^{\infty} b_n \exp[i(n\omega_c + k_2 v_{0z} - \omega)t]\,,$$

$$(15c)\qquad (D)D\zeta = \sum_{-\infty}^{\infty} c_n \exp[i(n\omega_c + k_2 v_{0z} - \omega)t]\,.$$

The solution of (15a) and (15b) is given by

$$(16)\qquad \left.\begin{matrix} D\psi \\ D\eta \end{matrix}\right\} = \exp[i\omega_c t]\int^t \exp[-2i\omega_c t]\,\mathrm{d}t \int^t \sum_{-\infty}^{\infty} \begin{matrix} a_n \\ b_n \end{matrix} \exp[i[(n+1)\omega_c + k_2 r_{0z} - \omega]t]\mathrm{d}t\,.$$

The solution of (15c) is given by

$$(17)\qquad D\zeta = \int^t \sum_{-\infty}^{\infty} c_n \exp[i(n\omega_c + k_2 v_{0z} - \omega)t]\,\mathrm{d}t\,.$$

What we have found here in integral form, are the perturbations in the v_x, v_y and v_z components of the electron velocities, a result of some interest. If the electrons in equilibrium are distributed according to some distribution function $f_0(v)$, its perturbation due to the perturbing electric field is given to first order by

$$(18)\qquad f_1 = \frac{\partial f_0}{\partial v_{0x}} D\chi + \frac{\partial f_0}{\partial v_{0y}} D\eta + \frac{\partial f_0}{\partial v_{0z}} D\zeta\,,$$

in a Taylor expansion.

From the maxwellian electromagnetic equations, we can derive the following equation

$$(19)\qquad \nabla(\nabla\cdot\boldsymbol{E}) - \nabla^2\boldsymbol{E} = -\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} - \frac{4\pi}{c^2}\frac{\partial\boldsymbol{J}}{\partial t}\,,$$

where

$$\boldsymbol{J} = -e\iiint \boldsymbol{v}\cdot f_1(\boldsymbol{r}, \boldsymbol{v}, t)\,\mathrm{d}v_x\,\mathrm{d}v_y\,\mathrm{d}v_z\,,$$

and where f_1 is given by (18).

From the three-component eq. (19), a fairly general dispersion equation for the coupled wave motion may be obtained. Instead of deriving a such general dispersion equation, we shall make some successive assumptions.

3. – Pure longitudinal oscillations.

It can easily be verified that when $k_1 = 0$, E_x and E_y are independent of E_z and may be put equal to zero. Consequently $D\chi = D\eta = 0$. In this case it is sufficient to study eq. (14) only,

$$(20) \qquad D(D\zeta) = -\frac{e}{m} E_{0z} \sum_{-\infty}^{\infty} i^n J_n(k_1 r) \exp[i\boldsymbol{k}\boldsymbol{r}_0 + in\varphi_0] \exp[i(n\omega_c + k_2 v_{0z} - \omega)t] .$$

When $k_1 = 0$,

$$J_n \equiv 0 \qquad J_0 = 1 \qquad (n = \pm 1, \pm 2, \ldots).$$

This yields

$$(21) \qquad D(D\zeta) = -\frac{e}{m} E_{0z} \exp[ik_2 z_0] \exp[i(k_2 v_{0z} - \omega)t] .$$

One integration yields

$$(22) \qquad D\zeta = -\frac{e}{m} E_{0z} \exp[ik_2 z_0] \frac{\exp[i(k_2 v_{0z} - \omega)t]}{i(k_2 v_0 - \omega)} ,$$

when $k_2 v_{0z} \neq \omega$.

When $k_2 v_{0z} - \omega \to 0$, we shall write the solution of (21) as

$$(23) \qquad D\zeta = -\frac{e}{m} E_{0z} \exp[ik_2 z] \cdot \lim_{T \to \infty} \frac{1}{2} \int_{-T}^{T} \exp[i(k_2 v_{0z} - \omega)t] \, dt =$$
$$= -\frac{e}{m} E_{0z} \exp[ik_2 z] \pi \delta(k_2 v_{0z} - \omega) .$$

The factor $\frac{1}{2}$ in front of the integral is due to the fact that the integration is performed over a double infinity range.

Combining this result with (22), we may write for the perturbation $D\zeta$

$$(24) \qquad D\zeta = -\frac{e}{i \cdot m} E_{0z} \exp[ik_2 x_0] \left[P \frac{\exp[i(k_2 v_{0z} - \omega)t]}{k_2 v_{0z} - \omega} + \pi i \, \delta(k_2 v_{0z} - \omega) \right] ,$$

where P stands for principal value and the meaning is the same as suggested by eq. (22). Since in eq. (18) the perturbations $D\chi$ and $D\eta$ equal zero in the

pure longitudinal case, the perturbation in the distribution function is given by

$$(25)\qquad \begin{cases} f_1 = \dfrac{\partial f_0}{\partial v_{0z}} \cdot D\zeta \,, \\ f_1 = \dfrac{\partial f_0}{\partial v_{0z}} \dfrac{ie}{m} E_{0z} \left[P \dfrac{\exp\left[i(k_2 z - \omega t)\right]}{k_2 v_{0z} - \omega} + \pi i \exp\left[ik_2 z_0\right] \delta(k_2 v_{0z} - \omega) \right] . \end{cases}$$

Since f_1 has a singularity for $v_{0z} = \omega/k_2$, we know *a priori* that this linear treatment is non-valid. Since, however, the distribution function is used for evaluating mean values as temperature, mean velocity, density, etc., we shall see that finite values for these quantities can be found. This is possible since the integral of $\delta(k_2 v_{z0} - \omega)$ over velocity space is finite. Let us evaluate the perturbation in electron density as an example. It is given by

$$(26)\qquad N_1 = \int\limits_{-\infty}^{\infty}\int\int f_1 \, \mathrm{d}v_{0x} \, \mathrm{d}v_{0y} \, \mathrm{d}v_{0z} = \int\limits_{-\infty}^{\infty} \bar{f}_1(v_{0z}) \, \mathrm{d}v_{0z} \,,$$

where

$$\bar{f}_1 = \int\limits_{-\infty}^{\infty}\int f_1 \, \mathrm{d}v_{0x} \, \mathrm{d}v_{0y} \,.$$

By the use of (25), we obtain for N_1

$$(27)\qquad N_1 = \frac{ie}{k_2 m} E_z \left[P \int\limits_{-\infty}^{\infty} \frac{\bar{f}_0'(v_{0z})}{v_{0z} - \omega/k_2} \, \mathrm{d}v_{0z} + \pi i \bar{f}_0' \left(\frac{\omega}{k_2} \right) \right]$$

If we equal this expression for N_1 to

$$N_1 = \frac{1}{4\pi e} ik_2 E_z \,,$$

obtained from $\mathrm{d}E_z/\mathrm{d}z = 4\pi e N_1$, we arrive at the following dispersion equation:

$$(28)\qquad 1 = \frac{4\pi e^2}{mk^2} \left[P \int\limits_{-\infty}^{\infty} \frac{\bar{f}_0'(v_{0z})}{v_{0z} - \omega/k_2} \, \mathrm{d}v_{0z} + \pi i \bar{f}_0' \left(\frac{\omega}{k_2} \right) \right] ,$$

where ω may be real as well as complex.

This equation which forms the basis for investigations of wave propagation, will now be studied in the case $k_2 \to 0$. We assume *a priori* that $\omega'/k^2 = \mathrm{Re}\,(\omega)/k^2 \to \infty$ and that $\gamma/k^2 = -\,\mathrm{Im}\,(\omega)/k_2 \to 0$.

By a series expansion in ω/k_z we arrive at equations which may be solved by the method of successive approximations. Up to first order in γ/k_z the results are

$$\omega'^2 = \omega_p^2 + 3\,\frac{\varkappa T}{m}\,k_z^2\,, \qquad \gamma = \frac{\omega_p^4 m}{2k_z^3 \varkappa T N_0}\,\bar{f}_0(\omega_p/k_z)\,. \tag{29}$$

Here $\varkappa =$ Boltzmann's constant, $T =$ temperature, $\omega_p =$ plasma frequency and

$$\bar{f}(v_{0z}) = N_0 \left(\frac{m}{2\pi\varkappa T}\right)^{\frac{1}{2}} \exp\left[-\frac{m}{2\varkappa T}\,v_{0z}^2\right].$$

The first of the eq. (29) is the result of VLASOV [1], and the second is the damping factor first calculated by LANDAU [2]. The latter is a direct consequence of the second term of the right-hand side of eq. (28), which in turn is due to the singular solution $D\zeta$ given by eq. (24).

From eq. (21) we see that electrons having a zero-order velocity $v_{0z} = \omega'/k_z$ will in our linear treatment experience an electric field which is constant in space. A consequence of this is a constant acceleration of the particles with velocities around ω'/k_z, and we may suspect that these electrons will deprive the electric field of its energy, which then becomes damped. In the next section, we shall justify this suspicion.

4. – Calculation of the energy absorption.

An expression for the mean value in space and time of the energy absorption of the electrons is

$$\Delta A = \int_{-\infty}^{\infty} \mathrm{d}v \langle \overline{f(v)(-e)Ev} \rangle\,, \tag{30}$$

where $\langle\ \rangle$ indicates mean value in time. The bar indicates mean value in space, $E = E_z$ and $v = v_z$ must be real. We also write $k = k_z$.

By a Taylor expansion we may write for the electron distribution function:

$$f(v) = f_0(v_0) + \frac{\mathrm{d}f_0}{\mathrm{d}v_0}\,\delta v_z + \ldots\,, \tag{31}$$

where

$$v = v_0 + \delta N_z\,, \qquad \delta v_z = \mathrm{Re}\,(\xi)\,.$$

By using the expressions (6) and (7), we easily find that the mean value

in space of $f(N)Ev$ is of second order and is equal to

$$f_0(v_0)\left(1-\frac{v_0^2}{\overline{u^2}}\right)E\delta v_z\,, \qquad \left(\overline{u^2}=\frac{\varkappa T}{m}\right),$$

where $\overline{u^2}$ is the mean square value of the velocity and $f_0(v_0)$ is the Maxwell distribution function. Since we have assumed $\omega'/k \gg \sqrt{\overline{u^2}}$ and also shall show that the main contribution to the absorption comes from a very small range around $v_0=\omega'/k$, 1 may be neglected compared to $v_0^2/\overline{u^2}$. Thus the integrand may be written

$$\overline{\langle f(v)Ev\rangle} = -\langle Ef_0(v_0)v_0^2\frac{m}{\varkappa T}\delta v_z\rangle\,,$$

which inserted into (5) yields

$$\Delta A = e\int_{-\infty}^{\infty}\mathrm{d}v\,\overline{\langle Ef_0(v_0)v_0^2\frac{m}{\varkappa T}\delta v_z\rangle}\,. \tag{33}$$

The real values of E and $\dot{\zeta}$ are found from (1) and (4) respectively and give

$$E = E_0\exp[-\gamma t]\cos(kz-\omega t)\,, \tag{34}$$

$$\delta v_z = \mathrm{Re}\,(\dot{\zeta}) = -\frac{e}{m}E_0\exp[-\gamma t]\,\frac{(kv_0-\omega)\sin(kz-\omega t)-\gamma\cos(kz-\omega t)}{(kv_0-\omega)^2+\gamma^2}\,, \tag{35}$$

where we *a priori* have introduced a damping factor γ by replacing ω with $\omega-i\gamma$. For the calculation of the absorption, this γ is almost irrelevant if the decay time $1/\gamma$ is much larger than the time over which we take the mean value.

By inserting (34) and (35) into (33) and taking the mean values, we may at last write

$$\Delta A = \frac{e^2}{2\varkappa T}\langle E_0^2\exp[-2\gamma t]\rangle\int_{-\infty}^{\infty}\mathrm{d}v_0 f_0(v_0)v_0^2\,\frac{1}{\gamma(1+[(kv_0-\omega)/\gamma]^2)}\,.$$

When $\gamma/k\to 0$, the function

$$\frac{1/\pi\gamma}{1+(k^2/\gamma^2)(v-\omega/k)^2}$$

is a representation of Dirac's δ-function, and the integral can easily be evaluated.

If for a certain k, γ becomes small enough but finite, the function $f_0(v_0)v_0^2$ may be put outside the integral with $v_0 = \omega/k$. Then the integration is straightforward. The result is

$$\Delta A = \frac{e^2}{2\varkappa T} E_0^2 \exp[-2\gamma t] f_0\left(\frac{\omega}{k}\right)\frac{\omega^2}{k^3}\pi .$$

As is easily verified, the damping coefficient of the electric wave E may be defined by the following expression:

$$2\gamma = \frac{\Delta A}{E^2/\gamma\pi} = \frac{-(\partial/\partial t)E^2\gamma\pi}{E^2\gamma\pi} .$$

By this definition we obtain for γ:

$$\gamma = \frac{\pi\omega_p^4 m}{\varkappa T k^3 N_0} f_0\left(\frac{\omega_p}{k}\right) . \tag{36}$$

Here we have put $\omega = \omega_p$ which is legitimate for very small k.

When compared to the damping coefficient found by LANDAU [2], there lacks a factor 2 in the denominator of (36). This discrepancy is most probably due to the two quite different ways of calculation.

5. – Discussion of the results of Sect. 4.

Judging from the present calculations we could say that the Landau damping might be due to an energy absorption of the electrons having the same or nearly the same velocity as the phase-velocity of the wave. These electrons experience a constant electric field and consequently become accelerated. But in order that the theory shall be consistent with the linearization condition $\zeta \ll \lambda/2\pi$, the electric field must be essentially damped in a time T_0 comparable with the time t needed for a resonant particle to traverse a lenght $\zeta \sim \lambda/2\pi$. For this to be the case, we must at least have

$$|\zeta_{\max}| = \frac{e}{m} E_0 \frac{1}{\gamma^2} < \frac{\lambda}{2\pi} \qquad \text{or} \qquad \gamma^2 > \frac{e}{m} kE_0 , \tag{37}$$

$|\zeta_{max}|$ is found from eq. (21) by integrating twice for electrons with velocity $v_0 = \omega/k$, and taking the real value.

If the condition (37) is not satisfied, the resonant electrons will in reality not experience a constant field (only in this linear theory), but an electric field which varies sinusoidally in space. Sinusoidally only as far as the field is not deformed by these electrons.

If the latter may be assumed, the electrons become trapped, and the mean value of the absorption will be zero.

However, the electric field will probably be deformed by the electrons becoming trapped and one has to include non-linear terms in order to account for this phenomenon.

In conclusion one may say that the Landau damping is real only for waves which satisfy the very strict linear condition (37). Waves which do not obey this condition, one may call non-linear. The field then contains enough energy to put into the « resonant electrons » so that these become trapped. These electrons will then probably not give rise to any damping of the wave.

It has however been asserted (4) that the Landau damping is due to a diffusion in configuration space, and the present calculations may be given an alternative interpretation.

Let us assume an electric wave, consisting of many waves with the same wave number k but with different frequencies ω around ω_p. The phase-velocity of the electric wave will then have values around ω_p/k, and electrons with these velocities will be resonant electrons in this case as well.

Since the electric wave-vector is damped because of diffusion in phase-space, the electric energy density is also damped. The lost energy must however be get rid of, and the calculations given show that this energy is transferred to the resonant electrons. Calculation of the energy absorption yields thus another way of finding the damping coefficient.

6. – Oscillations in the presence of an external magnetic field H_0.

We shall now assume both k_1 and $k_2 \neq 0$ and solve eq. (15) by the same method as applied in the case $k_1 = 0$, *i.e.* with the use of δ-functions. The solutions of (15*a*) and (15*b*) are given by the expressions

$$\left.\begin{matrix} D\chi \\ D\eta \end{matrix}\right\} = \exp\left[i\omega_r t\right] \int^t \exp\left[-2i\omega_r t\right] \mathrm{d}t \int^t \sum_{-\infty}^{\infty} \begin{matrix} a_n \\ b_n \end{matrix} \exp\left[i[n_0]t\right] \mathrm{d}t \,. \tag{38}$$

$$D\zeta = \sum_{-\infty}^{\infty} c_n \left(P \frac{\exp\left[i[n_0]t\right]}{i[n_0]} + \pi\delta[n_0]t \right). \tag{39}$$

Here

$$[n_0] = n\omega_0 + k_z v_{0z} - \omega \,.$$

We also introduce the following abbreviations

$$\begin{cases} [n_-] = (n-1)\omega_c + k_z v_{0z} - \omega \,, \\ [n_+] = (n+1)\omega_0 + k_z v_{0z} - \omega \,. \end{cases}$$

With the use of this notation, the expressions (38) and (39) can be written, after the integrations have been performed,

$$(40) \quad \left.\begin{matrix} D\chi \\ D\eta \end{matrix}\right\} = -\sum_{-\infty}^{\infty} a_n \left[P \frac{\exp[i[n_0]t]}{[n_-][n_+]} + \pi i \left(\frac{\exp[i\omega_c t]}{[n_-]} \delta[n_-] - \frac{\exp[-i\omega_c t]}{2\omega_c} \delta\, n_+] \right) \right]$$

$$(41) \quad D\zeta = -i \sum_{-\infty}^{\infty} c_n \left[P \frac{\exp[i[n_0]]}{[n_0]} + i\pi\delta[n_0] \right],$$

Now

$$(42) \quad f_1 = \frac{\partial f_0}{\partial v_{0x}} D\chi + \frac{\partial f_0}{\partial v_{0y}} D\eta + \frac{\partial f_0}{\partial v_{0z}} D\zeta \,.$$

Before we substitute the expressions (40) and (41) into (42), we introduce cylinder co-ordinates in the velocity space:

$$\begin{cases} v_{0x} = \varrho \cos\alpha = -\varrho \sin\varphi \,, \\ v_{0y} = \varrho \sin\alpha = \quad \varrho \cos\varphi \,, \\ v_{0z} = v_{0z} \,, \\ \varrho^2 = x_x^2 + v_y^2 \,, \end{cases}$$

(see Fig. 1). Then

$$\begin{cases} \dfrac{\partial f_0}{\partial v_{0x}} = -\sin\varphi \, \dfrac{\partial f_0(\varrho_z v_{0z})}{\partial \varrho} \,, \\ \dfrac{\partial f_0}{\partial v_{0y}} = \cos\varphi \, \dfrac{\partial f_0}{\partial \varrho} \,, \\ \dfrac{\partial f_0}{\partial v_{0z}} = \dfrac{\partial f_0}{\partial v_{0z}} \,. \end{cases}$$

The perturbation f_1 in the distribution function then becomes

$$
\begin{aligned}
(43)\qquad f_1 = {} & \frac{e}{2m}\frac{\partial f_0}{\partial \varrho}\exp\left[-ik_1 r\cos\varphi\right]\sum_{-\infty}^{\infty} i^n J_n(k_1 r)\cdot \\
& \cdot\left[P\frac{F_1\exp\left[i(n-1)\varphi\right]}{[n_-]} - P\frac{F_2\exp\left[i(n+1)\varphi\right]}{[n_+]}\right] + \\
& + i\frac{e}{m}\frac{\partial f_0}{\partial v_{0z}}\exp\left[-ik_1 r\cos\varphi\right]\sum_{-\infty}^{\infty} i^n J_n(k_1 r) P\frac{E_z\exp\left[in\varphi\right]}{[n_0]} + \\
& + \frac{\pi i e}{4m\omega_c}\sum_{-\infty}^{\infty}\exp\left[i\boldsymbol{k}\cdot\boldsymbol{r}_0 + in\varphi_0\right]\cdot i^n J_n\frac{\partial f_0}{\partial \varrho}\cdot \\
& \cdot\left[F_{10}[n_+]\,\delta[n_-]\exp\left[i\omega_c t - i\varphi\right] + F_{20}[n_-]\,\delta[n_+]\exp\left[-i\omega_c t + i\varphi\right]\right] + \\
& + \frac{\partial f_0}{\partial v_{0z}}\sum_{-\infty}^{\infty} -\frac{e}{m}\exp\left[i\boldsymbol{k}\boldsymbol{r}_0 + in\varphi_0\right] i^n J_n(k_1 r) E_{0z}\pi\,\delta[n_0]\,.
\end{aligned}
$$

where

$$
\begin{cases}
F_1 = F_{10}\exp\left[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\right] = E_x + iE_y\,, \\
F_2 = F_{20}\exp\left[i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\right] = E_x - iE_y\,.
\end{cases}
$$

This expression for f_1 is analogous to the f_1 in eq. (25) evaluated in the special case $H_0 = 0$, or what amounts to the same, $k_1 = 0$. In spite of the singularity there for $v_{0z} = \omega/k_2$, the distribution function could be used for calculations of mean values. Now we have singularities for

$$
v_{0z} = \frac{\omega - m\omega_c}{k_2}\,,
$$

where m is an integer.

We will now investigate whether this last derived f_1, eq. (43), can be used for evalutation of mean values. The perturbation N_1 of the electron density N_0, we find to be

$$
\begin{aligned}
(44)\qquad N_1 = {} & P\iiint f_1\cdot\varrho\,\mathrm{d}\varrho\,\mathrm{d}v_z\,\mathrm{d}\varphi - \frac{\pi e N_0}{2k_2 T}\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}}\iint\sum_{-\infty}^{\infty}\exp\left[-\frac{m}{2T}\varrho^2\right] J_n\exp[-k_1 r\cos\varphi] i^n\cdot \\
& \cdot\left[i\,F_1\exp\left[i(n-1)\varphi - \frac{m}{2T}v_-^2\right] - iF_2\cdot\exp\left[i(n+1)\varphi - \frac{m}{2T}v_+^2\right] - \right. \\
& \left. - \frac{2}{k_2}E_z(\omega - n\omega_c)\exp\left[-\frac{m}{2T}v_0^2 + in\varphi\right]\right]\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi\,,
\end{aligned}
$$

where we have used the equilibrium distribution function

$$
f_0(\varrho, v_{0z}) = N_0\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}}\exp\left[-\frac{m}{2T}(\varrho^2 + v_{0z}^2)\right],
$$

and the notation

$$\left.\begin{matrix} v_- \\ v_+ \end{matrix}\right\} = \frac{\omega - (n \mp 1)\omega_c}{k_z}, \qquad v_0 = \frac{\omega - n\omega_c}{k_z}.$$

In the case $ck \gg (\omega_p^2 + \omega_c^2)^{\frac{1}{2}}$, the coupling between the longitudinal and transversal modes is small (see BERNSTEIN [6]), and E_r may be put equal to zero when studying pure longitudinal waves.

Then the first term in eq. (44) can be written

$$(45) \qquad \iiint \sum_{-\infty}^{\infty} \frac{ie}{m} \exp\left[-ik_1 r \cos\varphi\right] i^n J_n \cdot \left[\frac{\partial f_0}{\partial \varrho} \frac{n\omega_c}{k_1 \varrho} E_x + \frac{\partial f_0}{\partial v_{0z}} E_z\right] P \frac{\exp\left[in\varphi\right]}{[n_0]} \varrho \, \mathrm{d}\varrho \, \mathrm{d}\varphi \, \mathrm{d}v_{0z},$$

where the following relation is utilized,

$$(46) \qquad J_{n+1} + J_{n-1} = \frac{2nJ_n(k_1 r)}{k_1 r}.$$

The integration over φ can be performed if we make use of the identity

$$(47) \qquad \int_0^{2\pi} i^n \exp\left[-ik_1 r \cos\varphi + in\varphi\right] \mathrm{d}\varphi = 2\pi J_n(k_1 r).$$

The eq. (45) may then be written

$$(48) \qquad i \frac{2\pi e}{m} P \iint \frac{J_n^2}{[n_0]} \left[\frac{\partial f_0}{\partial \varrho} \frac{n\omega_c}{k_1 \varrho} E_x + \frac{\partial f_0}{\partial v_{0z}} E_z\right] \varrho \, \mathrm{d}\varrho \, \mathrm{d}v_{0z}.$$

The second term of eq. (44), due to the δ-functions, can be shown by some calculations to yield

$$(49) \qquad \pi \frac{2\pi e N_0}{k k_z T} \left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} E_\omega \int \exp\left[-\frac{m}{2T}(\varrho^2 + v_0^2)\right] J_n^2 \varrho \, \mathrm{d}\varrho,$$

when

$$(50) \qquad f_0 = N_0 \left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} \exp\left[-\frac{m}{2T}(\varrho^2 + v_{0z}^2)\right].$$

Further in the pure longitudinal case

$$\frac{E_x}{E} = \frac{k_1}{k}\frac{E_z}{E} = \frac{k_2}{k}, \qquad E_x^2 + E_z^2 = E^2. \tag{51}$$

These relations and (47) as well, are used in order to arrive at (49).

When we insert (48) and (49) into (44) and put his expression for N_1 equal to that calculated from the Maxwell's divergence equation, we obtain the following dispersion equation

$$-1 = \frac{2\pi m\omega_0^2}{k^2 T}\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} \sum_{-\infty}^{\infty} \int_0^{\infty} \varrho\, \mathrm{d}\varrho J_n^2(k_1 r) \exp\left[-\frac{m}{2T}\varrho^2\right] \cdot \tag{52}$$

$$\cdot \left[P\int_{-\infty}^{\infty} (n\omega_c + k_2 v_{0z}) \exp\left[-\frac{m}{2T} v_{0z}^2\right] \frac{\mathrm{d}v_{0z}}{[n_0]} + \pi i \frac{\omega}{k_2} \exp\left[-\frac{m}{2T} v_0^2\right]\right].$$

Before giving a short discussion of this equation, we point out the fact that the two terms in the bracket can be found in another way. We have performed the integration along the real axis and used Dirac's δ-function, but the result is the same if we instead deform the integration path and integrate along C ([8]) (see Fig. 2).

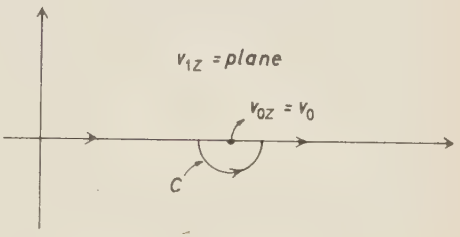

Fig. 2.

The path C consists of the real axis from $-\infty$ to $v_0 - \varepsilon$ and from $v_0 + \varepsilon$ to ∞, where ε is a small quantity which we let go to zero, plus the semicircle around the singularity $v_{0z} = (\omega - n\omega_c)/k_2 = v_0$.

We thus have

$$\int_C \frac{(n\omega_c + k_2 v_{0z}) \exp\left[-(m/2T) v_{0z}^2\right]}{n\omega_c + k_2 v_{0z} - \omega} \mathrm{d}v_{0z} = \tag{53}$$

$$= P\int_{-\infty}^{\infty} \frac{(n\omega_c + k_2 v_{0z}) \exp\left[-(m/2T) v_{0z}^2\right]}{[n_0]} \mathrm{d}v_{0z} + \pi i \frac{\omega}{k_2} \exp\left[-\frac{m}{2T} v_0^2\right].$$

The path C, the integration along which gives our result, eq. (52), represents the change in the integration path which Landau undertook in his investi-

([8]) A. G. Sitenko and K. N. Stepanov: *Soviet Phys. JETP*, **4**, 512 (1957).

gation of oscillations in a non-magnetic plasma. We mentioned this in the introduction and the consequences as well, namely the Landau damping. Also in the magnetic case this path deformation gives rise to a Landau damping, as long as the wave propagation is not transverse to the external field H_0, *i.e.* $k_2 \neq 0$. This damping has been thoroughly studied by SITENKO and STEPANOV ([8]) later also by BERNSTEIN ([6]).

When $k_2 = 0$, and at the same time $\omega \neq \omega_c$, the last term of eq. (53), which gives rise to the damping, is zero; consequently also the damping.

If, however, $k_2 \to 0$ and $\omega = n\omega_c$, the last term grows, and as a result of this linear investigation one may say that the « Landau damping » is extremely large for frequencies around multiples of the gyration frequency ω_c. This extreme damping constitutes the gaps first found by GROSS ([5]).

In the non-magnetic case the distribution function could be used for calculating mean values in spite of the singularity, since this disappeared by the integration. The same is true in the magnetic case as well, but after the integration, we have the additional singularity for $k_2 = 0$ and $\omega = n\omega_c$. When this condition is fulfilled, the density is singular and the linear treatment breaks down. To calculate the width of the frequency gaps is the same as to investigate how close to $n\omega_c$ one may go with ω and still keep the linear fluctuations within reasonable limits.

7. – The case $k_2 = 0$, $\omega \neq n\omega_c$ and the width of the frequency gaps.

The dispersion equation (52), when $k_2 = 0$ and the at same time $\omega \neq n\omega_c$, is found to be, after some calculations,

$$-1 = \frac{m^2\omega_c\omega_p^2}{k^2 T^2 \omega} \int_0^\infty \varrho \, \mathrm{d}\varrho \exp\left[-\frac{m}{2T}\varrho^2\right] \sum_{-\infty}^{\infty} \frac{n^2 J_n^2}{n - \omega/\omega_c} \,. \tag{54}$$

For small arguments of $J_n(kr)$ we have approximately

$$J_n^2(k\,r) = J_n^2\left(\frac{k\varrho}{\omega_c}\right) = \left(\frac{k^n \varrho^n}{\omega_c^n 2^n n!}\right)^2 \qquad n = 1, 2, \ldots .$$

With this approximation, and using the formula

$$\int_0^\infty \varrho^{2n+1} \exp\left[-b\varrho^2\right] \mathrm{d}\varrho = \frac{n!}{2b^{n+1}} \,,$$

we get for (54)

$$\frac{k^2 T}{m\omega_p^2} = \frac{k^2}{2b\omega_c^2(p^2-1)} + \frac{k^4}{2b^2\omega_c^4(p^2-4)} + \cdots .$$

Here $p = \omega/\omega_c$, $b = m/2T$.

When $1 < p < 2$ the first term on the right-hand side is positive, while the second and following terms are negative. For values of p_1 very close to unity, the first term is large and positive, and for values of p close to 2 the second term is large and negative. The contribution from the other terms is small. Since the left-hand side is always positive, we draw the conclusion that for frequencies ω in a region close to 2, the dispersion equation cannot be satisfied. The width of this forbidden region can be found to be

$$\Delta\omega = \frac{3}{4}\frac{Tk^2}{m\omega_c^2} .$$

For a more extensive treatment, see GROSS ([5]), BERNSTEIN ([6]), or KILDAL ([9]).

8. – Kinetic contra hydromagnetic description.

The singularities here studied do not appear when using the hydromagnetic description. Thus the Landau damping and the Gross gaps do not appear.

The reason for this is the following.

In a hydromagnetic description, the electrons are motionless in equilibrium, the thermal velocity being taken into account by the temperature T. Thus, in the non-magnetic case, no electrons have the velocity ω/k.

Consequently, we have no Landau damping since this damping is due to just those electrons.

Further, in the magnetic case, no electrons rotate in equilibrium when using the hydromagnetic model. Therefore the wave frequency is never in phase with the gyration frequency.

Since the gaps are due to a breakdown of the linear treatment, one cannot be sure that they are real. To find this out, one should take into account non-linear terms. For the case of Landau damping, in a non-magnetic plasma, BERNSTEIN *et al.* ([10]) have started a non-linear investigation.

([9]) A. KILDAL: *The Institute of Theoretical Astrophysics*, report n. 9, *On Oscillations in an Electron Plasma* (1959).

([10]) I. B. BERNSTEIN, J. M. GREENE and M. D. KRUSKAL: *Phys. Rev.*, **108**, 546 (1957).

When comparing the hydromagnetic and kinetic treatment of plasma oscillations, one finds conformity within the same order of approximation when k is small, except for the Landau damping and Gross gaps. As for the gaps the linear treatment breaks down and the comparison is no longer within the same order of approximation.

A plausible reason for the real existence of the gaps, is that they are due to an energy absorption. On a later occasion we hope to come back to this problem and the Landau damping as well.

RIASSUNTO (*)

Esponiamo una teoria lineare delle oscillazioni della forma $\exp[i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)]$ in un plasma di elettroni privo di collisioni in un campo magnetico esterno H_0. La teoria è basata sulle equazioni del moto di ogni particella. Studiamo solo il caso $\boldsymbol{k}\to 0$ e le singolarità che allora appaiono, vengono trattate facendo uso delle funzioni δ. Troviamo le espressioni per le perturbazioni $\delta\boldsymbol{v}$ della velocità degli elettroni. Con uno sviluppo di Taylor in $\delta\boldsymbol{v}$ valutiamo la perturbazione della funzione di distribuzione ed anche la perturbazione N_1 della densità. Quest'ultima quantità soddisfa anche all'equazione della divergenza. Per eliminazione otteniamo un'equazione della dispersione. Nel caso $\boldsymbol{k}_{\parallel}\boldsymbol{H}_0$, gli elettroni con velocità uguale alla velocità di fase subiscono un campo elettrico costante. Venendo così accelerati, essi privano il campo della sua energia. Calcoliamo questo assorbimento di risonanza e lo smorzamento concomitante si accorda allo smorzamento di Landau. Poi studiamo i gap nella frequenza che appaiono nel caso $\boldsymbol{k}\cdot\boldsymbol{H}_0=0$.

(*) *Traduzione a cura della Redazione.*

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Partial Wave Dispersion Relations in $\bar{K}$-Nucleon Absorption - I.

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Summary. — Using the double dispersion relations, the singularities of partial wave amplitudes for transitions between the coupled channels $\bar{K}+\mathcal{N}$, $\pi+\Sigma$ and $\pi+\Lambda$, are found. A formalism for solving the dispersion relations, using the unitarity condition, is written down according to the proposal of Bjorken.

1. – Introduction.

In the study of pion-nucleon scattering it has proved convenient to deduce, from the double dispersion relations, the analytic properties of the individual partial waves and to use these properties to set up dispersion relations directly in each partial wave of interest ([1,2]).

MacDowell was the first to study the analytic properties of partial waves, using the Mandelstam relations, and he did so in the context of K-nucleon scattering ([3]). In this paper we shall be particularly interested in $\bar{K}$-nucleon absorption, but unitarity couples the processes, $\bar{K}+\mathcal{N}\to\bar{K}+\mathcal{N}$, $\bar{K}+\mathcal{N}\to\pi+Y$ and $\pi+Y\to\pi+Y$, so that it will be necessary to examine the analytic properties of all these amplitudes.

In Section 2 a general discussion of the singularities in the partial wave amplitudes is given and the unitarity condition is written down. The singularities are, as usual, divided into kinematic and dynamical singularities and in Section 3 the latter are discussed in detail and evaluated as far as possible.

([1]) S. C. Frautschi and J. D. Walecka: *Phys. Rev.* (to be published).
([2]) J. Hamilton and T. D. Spearman: *Ann. Phys.* (to be published).
([3]) S. W. MacDowell: *Phys. Rev.*, **116**, 774 (1959).

The method of solution of the dispersion relations, taking the dynamical singularities as known is discussed in Section **4**. In a following paper (II) the results of the present paper will be used to assess the influence of the various dynamical singularities by noting some solutions under particular assumptions. The coupling constants and other parameters of the dynamical singularities are (except for certain ones given by pion-nucleon scattering) to be determined by a comparison of the solutions of these and like relations with the results of K-meson experiments. Unfortunately the experimental results are not yet adequate for such a purpose. What can be attempted is the distinction of the effect of important singularities such as the effect of a π-π singularity on K^--p scattering ([4]).

It is assumed that the parity of the K-meson is odd with respect to both the $\Lambda\bar{\mathcal{N}}$ and $\Sigma\bar{\mathcal{N}}$ pair. Attention will be confined to s-state, that is low-energy, absorption though most of the work can readily be carried over to p-waves. It will be noted that in considering the coupling of amplitude by unitarity we have neglected all but two-particle states. This neglect, while probably a valid approximation in pion-nucleon scattering below 300 MeV, is more serious here. Even at the threshold energy of $\bar{\mathrm{K}}+\mathcal{N} \rightleftharpoons \pi+\mathrm{Y}$ the process $\bar{\mathrm{K}}+\mathcal{N} \rightleftharpoons \rightleftharpoons \Lambda+2\pi$ is energetically possible. The importance of this effect requires examination and it will be assessed in another paper.

2. – Partial wave singularities. Unitarity condition.

Consider the process in which a baryon and a meson produce a baryon and a meson where the initial and final particles may be the same, or different:

$$\mathrm{B}_1+\mu_1 \to \mathrm{B}_2+\mu_2 .$$

Let B_1, B_2 also denote the masses of initial and final baryons and μ_1, μ_2 the masses of the initial and final mesons; let p_1, p_2 be the corresponding baryon 4-momenta and k_1, k_2 the corresponding meson 4-momenta so that

$$p_1+k_1=p_2+k_2 .$$

Then our notation is

$$(1)\qquad \begin{cases} s=-(p_1+k_1)^2 , \quad u=-(p_1-k_2)^2 , \quad t=-(k_1-k_2)^2 , \\ s+u+t=\sum m_i^2=B_1^2+\mu_1^2+B_2^2+\mu_2^2 . \end{cases}$$

([4]) F. Ferrari, G. Frye and M. Pusterla: *Phys. Rev. Lett.*, **4**, 615 (1960).

The scattering amplitude, in a state of isotopic spin T, is given by

$$-A^T + \tfrac{1}{2} i\gamma\cdot(k_1+k_2)B^T ,$$

where A^T, B^T are Lorentz invariant functions of s, u, t. The functions f_1^T, f_2^T are defined by

$$(2)\qquad \begin{cases} 4\pi f_1^T = \dfrac{\{[(W+B_1)^2-\mu_1^2][(W+B_2)^2-\mu_2^2]\}^{\frac{1}{2}}}{4W^2}[A^T+(W-M)B^T], \\ 4\pi f_2^T = \dfrac{\{[(W-B_1)^2-\mu_1^2][(W-B_2)^2-\mu_2^2]\}^{\frac{1}{2}}}{4W^2}[-A^T+(W+M)B^T], \end{cases}$$

where $M=(B_1+B_2)/2$ and $W=\sqrt{s}$ is the total energy in the centre of mass system.

If θ is the scattering angle in the centre of mass system and $x=\cos\theta$, we can express f_1^T, f_2^T as functions of s, x. Then

$$(3)\qquad g_{l\pm}^T(s) = \tfrac{1}{2}\int_{-1}^{1}(f_1^T P_l(x) + f_2^T P_{l\pm}(x))\,\mathrm{d}x ,$$

where $g_{l\pm}^T(s)$ is the partial wave amplitude of orbital angular momentum l and total angular momentum $l\pm\frac{1}{2}$. We denote the S-state partial wave simply by g^T so that

$$(4)\qquad g^T(s) = \tfrac{1}{2}\int_{-1}^{1}(f_1^T + x f_1^T)\,\mathrm{d}x .$$

(This definition of partial wave amplitude corresponds to scattering amplitudes of the form $g_{l\pm}^T = \sin\delta\exp[i\delta]/k$.)

The A^T and B^T both obey Mandelstam relations of the form

$$(5)\qquad A^T = P + \int_{s_0}^{\infty}\mathrm{d}s'\int_{t_0}^{\infty}\mathrm{d}u'\,\frac{\varrho_{12}^T(s',u')}{(s'-s)(u'-u)} + \int_{u_0}^{\infty}\mathrm{d}u'\int_{t_0}^{\infty}\mathrm{d}t'\,\frac{\varrho_{23}^T(u',t')}{(u'-u)(t'-t)} + \int_{t_0}^{\infty}\mathrm{d}t'\int_{s_0}^{\infty}\mathrm{d}s'\,\frac{\varrho_{31}^T(t',s')}{(t'-t)(s'-s)} ,$$

where P represents the pole terms. From the cuts and poles in A^T, B^T the singularities in f_1^T, f_2^T can be deduced using eq. (2) and (4). These singularities are of three kinds:

i) we shall call *dynamical singularities* those singularities due to the pole terms or the cuts in u, t in the Mandelstam relations for A^T, B^T. The discontinuities across these singularities are taken as known or unknown parameters of the theory;

ii) the remaining cuts in (5) give rise to *kinematical singularities* in g^T. These are the s-cuts in the Mandelstam relations for A^T, B^T and include the physical region for the reaction $B_1+\mu_1 \to B_2+\mu_2$;

iii) there is a pole at $s=0$ and a cut for $-\infty \leqslant s \leqslant 0$ from the multiplying factors in (2).

It appears that there may also be an irrationality cut for $(B_1-\mu_1)^2 \leqslant s \leqslant (B_2-\mu_2)^2$ and $(B_2+\mu_2)^2 \leqslant s \leqslant (B_1+\mu_1)^2$ (if for example $(B_1-\mu_1)^2 < (B_2-\mu_2)^2 < (B_2+\mu_2)^2 < (B_1+\mu_1)^2$). If for example that branch is taken for which $W=\sqrt{s}>0$ for s real and positive, then $[(W+B_1)^2-\mu_1^2][(W+B_2)^2-\mu_2^2]>0$ and it can be seen in detail on using (4) that $\{[(W-B_1)^2-\mu_1^2]\cdot[(W-B_2)^2-\mu^2]\}^{\frac{1}{2}}$ does not occur in the partial wave amplitude (4).

The dynamical singularities will be fully investigated in the next section; the kinematical singularities are trivially given.

Consider the application of unitarity to the reaction $\overline{\mathrm{K}}+\mathcal{N} \to \pi+\mathrm{Y}$. We denote the $\overline{\mathrm{K}}\mathcal{N}$, $\pi\Sigma$ and $\pi\Lambda$ channels by 1, 2 and 3 respectively. In the $T=0$ state only channels 1 and 2 enter. For a particular angular momentum and isotopic spin there are at most three channels and the transition amplitude can be labelled g_{ij}^T $(i, j = 1, 2, 3)$ where the angular momentum labelling has been dropped and, for example, g_{11}^T is the transition amplitude for $\overline{\mathrm{K}}+\mathcal{N} \to \mathrm{K}+\mathcal{N}$. Time-reversal invariance ensures that the phases of the states may be chosen so that [5]

$$g_{ij}^T = g_{ji}^T . \tag{7}$$

Taking the approximation of two particle states only, in the unitarity condition, this may be written [4]

$$\mathrm{Im}\,(g^{-1})_{ij} = -k_i \theta_i \delta_{ij} , \tag{6}$$

where θ_i is a step function 0 or 1 according as $W=\sqrt{s}$ is less than or greater than the threshold energy in the i-th channel; k_i is the magnitude of the centre of mass momentum in the i-th channel.

[5] F. Coester: *Phys. Rev.*, **89**, 619 (1953).

3. – Analysis of partial wave singularities.

In this section we consider the kinematical and dynamical singularities of the partial wave amplitudes for the different processes. The most compli cated are the amplitudes g^T_{12}, g^T_{13} for

$$\overline{\mathrm{K}} + \mathcal{N} \to \pi + \Sigma$$

$$\overline{\mathrm{K}} + \mathcal{N} \to \pi + \Lambda$$

and these will be taken first. When particularization is made to a certain angular momentum, it will be the S-state amplitude that will be chosen. Much of what is said is applicable to any angular momentum state.

3·1. *Singularities in* g^T_{12}. – The kinematic singularities are obviously

$$\text{(7)} \qquad \begin{cases} T = 0: & (\mu + \Sigma)^2 \leqslant s \leqslant \infty \\ T = 1: & (\mu + \Lambda)^2 \leqslant s \leqslant \infty \end{cases}$$

with the notation that the symbol for the particle also denotes the mass, except that the mass of the pion is denoted by μ.

The pole terms in A^T, B^T for $\overline{\mathrm{K}} + \mathcal{N} \to \pi + \Sigma$ are given by

$$\text{(8)} \qquad \begin{cases} A^0: & \dfrac{2\Lambda - \Sigma - N}{s - \Lambda^2} \sqrt{\dfrac{3}{2}}\, g_{\Lambda \mathrm{K}} g_{\Lambda\pi} - \dfrac{\Sigma - N}{u - N^2} \sqrt{\dfrac{3}{2}}\, g_{\Sigma \mathrm{K}} g_{\mathcal{N}\pi}\,, \\ B^0: & -\dfrac{1}{s - \Lambda^2} \sqrt{6}\, g_{\Lambda \mathrm{K}} g_{\Lambda\pi} + \dfrac{1}{u - N^2} \sqrt{6}\, g_{\Sigma \mathrm{K}} g_{\mathcal{N}\pi}\,, \\ A^1: & \dfrac{\Sigma - N}{s - \Sigma^2} g_{\Sigma \mathrm{K}} g_{\Sigma\pi} - \dfrac{\Sigma - N}{u - N^2} g_{\Sigma \mathrm{K}} g_{\mathcal{N}\pi}\,, \\ B^1: & -\dfrac{1}{s - \Sigma^2} 2 g_{\Sigma \mathrm{K}} g_{\Sigma\pi} + \dfrac{1}{u - N^2} 2 g_{\Sigma \mathrm{K}} g_{\mathcal{N}\pi}\,. \end{cases}$$

In (8) $g_{\Lambda\pi}$, for example, denotes the coupling constant for the Λ K, $\mathcal{N}$ interaction. Using (2) and (4) it is found immediately that the S-state amplitude g^T_{12} contains no dynamical singularities coming from the first terms of the expressions (8) (that is the uncrossed poles of A^T, B^T).

There are also the singularities that arise from the crossed poles in A^T, B^T as given in (8). The singularities are given by the vanishing of $N^2 - u$ for

values of $x(\cos\theta)$ between -1 and $+1$. From eq. (1)

$$u = c - s - t\,, \tag{9}$$

$$\therefore N^2 - u = N^2 - c + s + t\,, \tag{10}$$

$$t = -(k_1 - k_2)^2 = \mu^2 + K^2 - 2k_{10}k_{20} + 2|\boldsymbol{k}_1||\boldsymbol{k}_2|x\,. \tag{11}$$

It is easily shown that

$$k_{i0} = \frac{W^2 - B_i^2 + \mu_i^2}{2W}\,, \qquad (i = 1, 2)\,, \tag{12}$$

$$|\boldsymbol{k}_i| = \{[(W + B_i)^2 - \mu_i^2][(W - B_i)^2 - \mu_i^2]\}^{\frac{1}{2}}/2W = \tag{13}$$
$$= \{[s - (B_i + \mu_i)^2][s - (B_i - \mu_i)^2]\}^{\frac{1}{2}}/2W\,.$$

So from eq. (11) we find

$$t = \mu^2 + K^2 - (s + N^2 - K^2)(s + \Sigma^2 - \mu^2)/2s + \tag{14}$$
$$+ \{[s - (N + K)^2][s - (N - K)^2][s - (\Sigma + \mu)^2[s - (\Sigma - \mu)^2]\}^{\frac{1}{2}}x/2s\,.$$

Thus from (10)

$$N^2 - u = a + xb\,, \tag{15}$$

where

$$a = N^2 - c/2 + s/2 - (N^2 - K^2)(\Sigma^2 - \mu^2)/2s\,, \tag{16}$$

$$b = \{[s - (N + K)^2][s - (N - K)^2][s - (\Sigma + \mu)^2][s - (\Sigma - \mu)^2]\}^{\frac{1}{2}}/2s\,. \tag{17}$$

Substituting the second terms of (8) into (4) and (2) and performing the integration over x, we obtain as the contribution to g_{12}^{T} from the pole terms:

$$\frac{C}{2b}\ln\frac{a+b}{a-b}\left\{\frac{W-\Sigma}{4\pi}\frac{\{[(W+N)^2-K^2][(W+\Sigma)^2-\mu^2]\}^{\frac{1}{2}}}{4W^2} - \right. \tag{18}$$
$$\left. - \frac{a}{b}\frac{W+\Sigma}{4\pi}\frac{\{[(W-N^2)-K^2][(W-\Sigma)^2-\mu^2]\}^{\frac{1}{2}}}{4W^2}\right\} + \text{« other terms »}\,,$$

where

$$C^0 = -\sqrt{6}\,g_{\Sigma K}g_{N\pi}\,, \qquad C^1 = -2g_{\Sigma K}g_{N\pi} \tag{19}$$

and where the « other terms » do not contain the logarithm and so do not contribute to the dynamical singularity, which comes wholly from the cut in

the logarithm [6]. The cut in the s-plane corresponding to the cut in the logarithm is given by those values of s for which $(a+b)/(a-b)$ is real and negative. This is given by the solutions of the quartic equation

$$a^2 = b^2 x^2 \tag{20}$$

for

$$0 \leqslant x^2 \leqslant 1 .$$

On solution of the equation it is found that the singularity is a cut along the following portion of the real axis:

$$\begin{cases} -\infty \leqslant s \leqslant 0 , \\ 5.248 \leqslant s \leqslant 6.290 , \end{cases} \tag{21}$$

where units in which the K-meson mass is unity are adopted: $K = 1$. From (18) the discontinuity across the cut, on going from the negative imaginary to the positive imaginary side $(g(s+i\varepsilon) - g(s-i\varepsilon))$ is given by:

$$-\frac{\pi i C^T}{b}\left\{\frac{(W-\Sigma)}{16\pi W^2}\left\{[(W+N)^2 - K^2][(W+\Sigma)^2 - \mu^2]\right\}^{-} - \right. \\ \left. - \frac{a}{b}\frac{(W+\Sigma)}{16\pi W^2}\left\{[(W-N)^2 - K^2][(W-\Sigma)^2 - \mu^2]\right\}^{\frac{1}{2}}\right\} . \tag{22}$$

In the method of solution of the dispersion relations (as given in Section 4 below) it is desirable to approximate the cut (21) on the positive real axis by a pole [1].

In that case the residue at the pole is given by

$$R = \int_{5.248}^{6.290} \mathrm{d}s \, \frac{C^T}{2b}\left\{\frac{(W-\Sigma)}{16\pi W^2}\left\{[(W+N)^2 - K^2][(W+\Sigma)^2 - \mu^2]\right\}^{\frac{1}{2}} - \right. \\ \left. - \frac{a}{b}\frac{(W+\Sigma)}{16\pi W^2}\left\{[(W-N)^2 - K^2][(W-\Sigma)^2 - \mu^2]\right\}^{\frac{1}{2}}\right\} = (.0025)\frac{C^T}{4\pi} , \tag{23}$$

and the pole may be taken at the point for which $a = 0$:

$$s = 5.821 .$$

---

[6] The fact that in (18) $(1/b)\ln(a+b)/(a-b)$ is an even function of b and that $\{[(W-N)^2 - K^2][(W-\Sigma)^2 - \mu^2]\}^{\frac{1}{2}}$ cancels with a factor of b shows that there is no additional cut for $(N-K)^2 \leqslant s \leqslant (\Sigma-\mu)^2$, $(\Sigma+\mu)^2 \leqslant s \leqslant (N+K)^2$.

There remain the dynamical singularities due to the cuts in u and t in the functions A^T, B^T

$$(N+\mu^2) = u_0 \leqslant u \leqslant \infty \,, \tag{25}$$

$$t_0 \leqslant t \leqslant W \,. \tag{26}$$

Take (25) first. Then *possible* singularities of g^T_{12} are given by the values of s for which u lies in the range (25) for values of x between -1 and 1. It is found as in the immediately preceding case of the « crossed pole » that these values of s are given by

$$a'^2 = b^2 x^2 \,, \tag{27}$$

where

$$a' = u' - c/2 + s/2 - (N^2 - k^2)(\Sigma^2 - \mu^2)/2s \,, \tag{28}$$

with $(N+\mu)^2 \leqslant u' \leqslant \infty$ and b given by (17).

It is found that solution of (27) gives a whole region of the s-plane besides a cut from $s = -\infty$ to ~ 1. So the « crossed cut » (25) gives a region of the s-plane where $g^T_{12}(s)$ may be non-continuable since this region is bounded by the curve α, shown in Fig. 1, which is a locus of branch points.

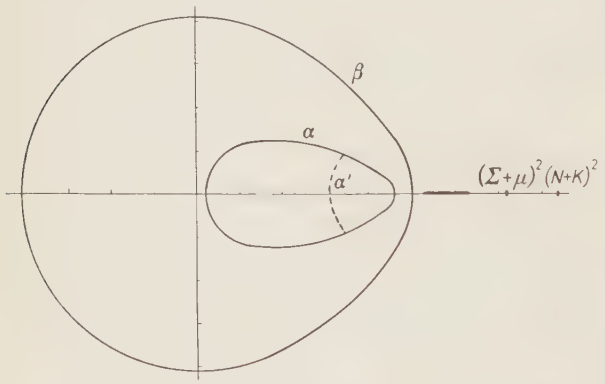

Fig. 1. – The curves α, β, α' together with the short cut on the real axis in the s-wave amplitude for $K+N \to \pi+\Sigma$. The axes are marked in units of K^2.

The exact analytic properties of $g^T_{12}(s)$ inside and on this curve α depend on the spectral functions ϱ of the Mandelstam representation (5) for A^T and B^T. It is compatible with the Mandelstam representation, for example, for g^T_{12} to be analytic inside and on α except for certain cuts and poles inside α; or to be analytic inside α, but with a cut along α (7). (Of course, from perturbation theory and unitarity arguments, we would expect α to be a curve of singularities, strictly speaking, so that the function cannot be extended through α. However, as noted below, an analytic continuation through α may be an allowable approximation in certain cases.)

(7) The curve α has been independently given by M. NAUENBERG (Cornell University thesis, unpublished).

$(N+\mu)^2 \leqslant u \leqslant \infty$ is the physical region for the crossed reaction

$$\pi + \mathcal{N} \to \mathrm{K} + \Sigma \,. \tag{29}$$

If the amplitudes for this reaction were known, then of course the analytic properties of $g^T_{12}(s)$ in and on α, from the « crossed cut », would be completely determined. Of course the amplitudes are not known; we may for interest assume that the reaction (29) has an amplitude in the unphysical region containing a factor $\delta(u - u_k)$ where u_k is the resonance energy for $\pi\mathcal{N}$ scattering. The corresponding singularity in $g^T_{12}(s)$ reduces to a cut $-\infty \leqslant s \leqslant 0$ and the cut α' shown in Fig. 1.

In the case of the dynamical singularities due to the cuts in t in A^T, B^T given by equation (26) an *anomalous threshold* ([8]) is involved. The normal threshold in eq. (26) would be given by $t_0 = (K+\mu)^2$, but instead ([8,9]) $t_0 = K^2 + \mu^2 + 2K\mu \cos 26°$. On the basis of perturbation theory the Mandelstam representation is equally valid with anomalous as with normal thresholds ([10]).

Possible singularities of g^T_{12} are given by those values of s for which

$$a''^2 = b^2 x^2 \,,$$

where b is given by (17) and

$$a'' = t' - e/2 + s/2 + (N^2 - k^2)(\Sigma^2 - \mu^2)/2s \,,$$

for $(K+\mu)^2 - (.202)K\mu \leqslant t' \leqslant \infty$.

It is found that there is a cut on the real axis from $-\infty$ to 0 and a region of possible non-continuability bounded by the curve β shown in Fig. 1. There is also a very short cut where β meets the positive real axis, due to the range of t between the anomalous and normal threshold.

3·2. *Singularities in* g^T_{13}. – The pole terms in A^T, B^T for $\overline{\mathrm{K}} + \mathcal{N} \to \pi + \Lambda$ are given by

$$\begin{cases} A^1\colon & \dfrac{2\Sigma - N - \Lambda}{s - \Sigma^2} \dfrac{1}{\sqrt{2}} g_{\Sigma \mathrm{K}} g_{\Lambda\pi} - \dfrac{\Lambda - N}{u - N^2} \dfrac{1}{\sqrt{2}} g_{\Lambda \mathrm{K}} g_{\mathcal{N}\pi} \,, \\ B^1\colon & -\dfrac{1}{s - \Sigma^2} \sqrt{2}\, g_{\Sigma \mathrm{K}} g_{\Lambda\pi} + \dfrac{1}{u - N^2} \sqrt{2}\, g_{\Lambda \mathrm{K}} g_{\mathcal{N}\pi} \,. \end{cases} \tag{8'}$$

([8]) R. Karplus, C. M. Sommerfield and E. H. Wichmann: *Phys. Rev.*, **111**, 1187 (1958); **114**, 376 (1959).

([9]) S. Mandelstam: *Phys. Rev. Lett.*, **4**, 84 (1960); R. E. Cutkosky: *Phys. Rev. Lett.*, **4**, 624 (1960).

([10]) R. J. Eden, P. V. Landshoff, J. C. Polkinghorne and J. C. Taylor: Cambridge preprint.

From which using (2) and (4) it follows that the S state amplitude contains no dynamical singularity from the uncrossed pole.

The dynamical singularities from (8′) consist of cuts along the following portions of the real axis

$$(21') \qquad -\infty \leqslant s \leqslant 0\,, \qquad 4.523 \leqslant s \leqslant 5.608\,.$$

The discontinuity across the cut $(g(+i\varepsilon) - g(s - i\varepsilon))$ is given by

$$(22') \qquad \frac{\pi i}{b}\sqrt{2}\, g_{\Sigma K}\, g_{\mathcal{N}\pi}\left\{\frac{W-\Lambda}{16\pi W^2}\left\{[(W+N)^2-K^2][(W+\Lambda)^2-\mu^2]\right\}^{\frac{1}{2}} - \right.$$
$$\left. - \frac{a}{b}\,\frac{W+\Lambda}{16\pi W^2}\left\{[(W-N)^2-K^2][(W-\Lambda)^2-\mu^2]\right\}^{\frac{1}{2}}\right\}.$$

Here a and b are given by eq. (16) and (17) with Σ replaced by Λ. In the case of approximating the shorter cut by a pole the residue at the pole is given by

$$(23') \qquad R = -\int_{4.523}^{5.608} \mathrm{d}s\, \frac{g_{\Lambda K}\, g_{\mathcal{N}\pi}}{\sqrt{2}\,b}\left\{\frac{W-\Lambda}{16\pi W^2}\left\{[(W+N)^2-K^2][(W+\Lambda)^2-\mu^2]\right\}^{\frac{1}{2}} - \right.$$
$$\left. - \frac{a}{b}\,\frac{W+\Lambda}{16\pi W^2}\left\{[(W-N)^2-K^2][(W-\Lambda)^2-\mu^2]\right\}^{\frac{1}{2}}\right\} = (-\,.0024)\,\frac{g_{\Lambda K}\, g_{\mathcal{N}\pi}\sqrt{2}}{4\pi}\,,$$

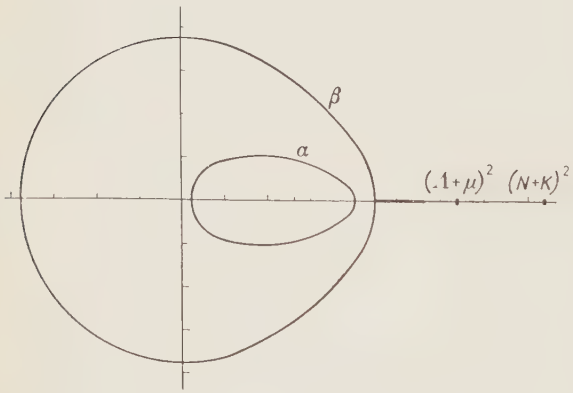

Fig. 2. – The curves α, β together with the short cut on the real axis in the s-wave amplitude $\mathrm{K}+\mathcal{N}\to\pi+\Lambda$. The axes are marked in units of K^2.

and the pole may be taken at $s = 5.095$.

The « singularities » due to the u and t cuts in A^T, B^T are similar to those in the $\overline{\mathrm{K}}+\mathcal{N}\to\pi+\Sigma$ case. (The cut in t is $(K+\mu)^2 - -(.002\,8)K\mu \leqslant t \leqslant \infty$, the anomalous threshold thus being very close to the normal threshold in this case.) The bounding curves α and β of the regions of possible non-continuability due to the u and t cuts respectively are shown in Fig. 2.

3·3. *Singularities in* g^T_{11}. – The partial wave singularities for the process $\overline{\mathrm{K}}+\mathcal{N}\to\overline{\mathrm{K}}+\mathcal{N}$, have been discussed by MacDowell [3] and no detailed de-

rivation will be presented here. In the invariant amplitudes A^T, B^T there are no crossed pole terms, and the contributions from the uncrossed pole terms are given by:

$$(33)\qquad \begin{cases} A^0: & \dfrac{\Lambda - N}{s-\Lambda^2}\, 2g^2_{\Sigma \mathrm{K}}\,, \\ B^0: & -\dfrac{1}{s-\Lambda^2}\, 2g^2_{\Lambda \mathrm{K}}\,, \\ A^1: & \dfrac{\Sigma - N}{s-\Sigma^2}\, 2g^2_{\Sigma \mathrm{K}}\,, \\ B^1: & -\dfrac{1}{s-\Sigma^2}\, 2g^2_{\Sigma \mathrm{K}}\,. \end{cases}$$

On application of eq. (2) and (4) it is found that no singularities result in the S-state amplitude.

The singularities due to the t cut $(4\mu^2 \leqslant t \leqslant \infty)$ in A^T, B^T are the following cuts in the s-plane.

$$(34)\qquad -\infty \leqslant s \leqslant 0\,,$$

$$(35)\qquad (\sqrt{N^2-\mu^2} - \sqrt{K^2-\mu^2})^2 \leqslant s \leqslant (\sqrt{N^2-\mu^2} + \sqrt{K^2-\mu^2})^2\,,$$

$$(36)\qquad |s| = N^2 - K^2\,.$$

The singularity due to the u cut $((N+k)^2 \leqslant u \leqslant \infty$ (in A^T, B^T is the following cut in the s-plane:

$$(37)\qquad T=0: \quad -\infty \leqslant s \leqslant 2(K^2+N^2) - (\Sigma+\mu)^2\,,$$

$$(38)\qquad T=1: \quad -\infty \leqslant s \leqslant 2(K^2+N^2) - (\Lambda+\mu)^2\,.$$

3·4. *Singularities in* g^T_{22}. – These singularities are for the process $\pi+\Sigma \to \to \pi+\Sigma$. The pole terms in A^T, B^T are:

$$(39)\qquad \begin{cases} A^0: & \dfrac{\Lambda-\Sigma}{s-\Lambda^2}\, 3g^2_{\Lambda\pi} + \dfrac{\Lambda-\Sigma}{u-\Lambda^2}\, g^2_{\Lambda\pi}\,, \\ B^0: & -\dfrac{1}{s-\Lambda^2}\, 3g^2_{\Lambda\pi} + \dfrac{1}{u-\Lambda^2}\, g^2_{\Lambda\pi} - \dfrac{1}{u-\Sigma^2}\, 2g^2_{\Sigma\pi}\,, \\ A^1: & -\dfrac{\Lambda-\Sigma}{u-\Lambda^2}\, g^2_{\Lambda\pi}\,, \\ B^1: & -\dfrac{1}{s-\Sigma^2}\, 2g^2_{\Sigma\pi} + \dfrac{1}{u-\Sigma^2}\, g^2_{\Sigma\pi} - \dfrac{1}{u-\Lambda^2}\, g^2_{\Lambda\pi}\,. \end{cases}$$

As before the S-state amplitude contains no singularities from the uncrossed poles. From the crossed poles in (39) we have the following cuts from the pole at $u = \Sigma^2$

$$-\infty \leqslant s \leqslant 0\,, \qquad \frac{(\Sigma^2-\mu^2)^2}{\Sigma^2} \leqslant s \leqslant \Sigma^2 + 2\mu^2\,, \tag{40}$$

and from the pole at $u = \Lambda^2$

$$-\infty \leqslant s \leqslant 0\,, \qquad \frac{(\Sigma^2-\mu^2)^2}{\Lambda^2} \leqslant s \leqslant 2\Sigma^2 - \Lambda^2 + 2\mu^2\,. \tag{41}$$

(These are associated with $g^2_{\Sigma\pi}$ and $g^2_{\Lambda\pi}$ respectively.)

The discontinuities across the cuts are given by

$$T = \begin{pmatrix}0\\1\end{pmatrix} : \frac{\pi i g^2_{\Lambda\pi}}{b} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \left\{ \frac{W + \Lambda - 2\Sigma}{16\pi W^2} [(W+\Sigma)^2 - \mu^2] - \frac{a}{b} \frac{W + 2\Sigma - \Lambda}{16\pi W^2} [(W-\Sigma)^2 - \mu^2] \right\}, \tag{42}$$

(cut (41))

$$T = \begin{pmatrix}0\\1\end{pmatrix} : \frac{\pi i g^2_{\Sigma\pi}}{b} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cdot \left\{ \frac{W - \Sigma}{16\pi W^2} [(W+\Sigma)^2 - \mu^2] - \frac{a}{b} \frac{W + \Sigma}{16\pi W^2} [(W-\Sigma)^2 - \mu^2] \right\}, \tag{43}$$

(cut (40))

where

$$a = Y^2 - \Sigma^2 - \mu^2 + s/2 - (\Sigma^2-\mu^2)^2/2s\,,$$

$$b = [s - (\Sigma+\mu)^2][s - (\Sigma-\mu)^2]/2s\,,$$

with $Y = \Lambda$ in eq. (42), $Y = \Sigma$ in eq. (43).

If we approximate the short cuts in (40) and (41) by poles the residue at these poles is given by:

$$\text{cut (41)},\ T = \begin{pmatrix}0\\1\end{pmatrix} : R = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{g^2_{\Lambda\pi}}{4\pi}\,(.0005)\,,$$

$$\text{cut (40)},\ T = \begin{pmatrix}0\\1\end{pmatrix} : R = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{g^2_{\Sigma\pi}}{4\pi}\,(.00007)\,.$$

The u cut ($T=0$: $(\Sigma+\mu)^2 \leqslant u \leqslant \infty$; $T=1$: $(\Lambda+\mu)^2 \leqslant u \leqslant \infty$) in A^s, B^T gives the following cuts in the S-state amplitude:

$$T = 0: \qquad -\infty \leqslant s \leqslant (\Sigma-\mu)^2\,, \tag{44}$$

$$T = 1: \qquad -\infty \leqslant s \leqslant 2\Sigma^2 + 2\mu^2 - (\Lambda+\mu)^2\,. \tag{45}$$

There is an anomalous threshold for the t-cut in A^T, B^T: $3\mu^2 \leqslant t \leqslant \infty$ (the normal threshold is at $t = 4\mu^2$). The t-cut gives the following cuts in the S-state amplitude:

$$-\infty \leqslant s \leqslant 0\,, \qquad |s| = \Sigma^2 - \mu^2\,,$$

$$\Sigma^2 - \tfrac{1}{2}\mu^2 - (\Sigma^2\mu^2 - \tfrac{3}{4}\mu^4)^{\frac{1}{2}} \leqslant s \leqslant \Sigma^2 - \tfrac{1}{2}\mu^2 + (\Sigma^2\mu^2 - \tfrac{3}{4}\mu^4)^{\frac{1}{2}}\,.$$

The last cut is wholly due to the occurrence of an anomalous threshold.

3·5. *Singularities in* g^T_{33}. g^T_{33} is the amplitude for the process $\pi + \Lambda \to \pi + \Lambda$ there is only one isospin channel, $T = 1$. The pole terms in A^1, B^1 are

$$\text{(47)} \qquad \begin{cases} A^1: & \dfrac{\Sigma - \Lambda}{s - \Sigma^2}\, g^2_{\Lambda\pi} + \dfrac{\Sigma - \Lambda}{u - \Sigma^2}\, g^2_{\Lambda\pi}\,, \\ B^1: & -\dfrac{1}{s - \Sigma^2}\, g^2_{\Lambda\pi} + \dfrac{1}{u - \Sigma^2}\, g^2_{\Lambda\pi}\,. \end{cases}$$

There are no singularities in the S-state amplitude g^1_{33} from the uncrossed poles, while we have the following cuts from the uncrossed poles:

$$\text{(48)} \qquad -\infty \leqslant s \leqslant 0\,, \qquad \frac{(\Lambda^2 - \mu^2)^2}{\Sigma^2} \leqslant s \leqslant 2\Lambda^2 + 2\mu^2 - \Sigma^2\,,$$

while the discontinuity across the cut is given by

$$\text{(49)} \qquad -\frac{\pi i g^2_{\Lambda\pi}}{b}\left\{\frac{W + \Sigma - 2\Lambda}{16\pi W^2}\,[(W + \Lambda)^2 - \mu^2] - \frac{a}{b}\,\frac{W + 2\Lambda - \Sigma}{16\pi W^2}\,[(W - \Lambda)^2 - \mu^2]\right\},$$

where

$$a = \Sigma^2 - \Lambda^2 - \mu^2 + s/2 - (\Lambda^2 - \mu^2)^2/2s\,,$$

$$b = [s - (\Lambda + \mu)^2][s - (\Lambda - \mu)^2]/2s\,.$$

If we approximate the short cut in (48) by a pole, the residue at the pole is given by

$$R = \frac{g^2_{\Lambda\pi}}{4\pi}\,(-.0005)\,.$$

The u cut $((\Lambda + \mu)^2 \leqslant u \leqslant \infty)$ in A^T, B^T gives the following cut in the S-state amplitude:

$$\text{(50)} \qquad -\infty \leqslant s \leqslant (\Lambda - \mu)^2\,.$$

While the t-cut, $4\mu^2 < t \leqslant \infty$, in A^T, B^T gives the following cuts in the S-state amplitude

$$-\infty \leqslant s \leqslant 0\,, \qquad |s| = \Lambda^2 - \mu^2\,. \tag{51}$$

3·6. *Singularities in* g^T_{23}. – g^T_{23} is the amplitude for the process $\pi + \Sigma \to \to \pi + \Lambda$ there is only one isospin channel, $T = 1$. This is an inelastic process, but the structure of the partial wave singularities bears more resemblance to that of the elastic processes rather than the inelastic processes. This is because the two mesons concerned are identical and the Σ and Λ masses do not differ greatly.

The pole terms in A^1, B^1 are

$$\begin{cases} A^1: & \dfrac{\Sigma - \Lambda}{s - \Sigma^2}\dfrac{1}{\sqrt{2}}\, g_{\Sigma\pi} g_{\Lambda\pi} + \dfrac{\Sigma - \Lambda}{u - \Sigma^2}\dfrac{1}{\sqrt{2}}\, g_{\Sigma\pi} g_{\Lambda\pi}\,, \\ B^1: & -\dfrac{1}{s - \Sigma^2}\sqrt{2} g_{\Sigma\pi} g_{\Lambda\pi} + \dfrac{1}{u - \Sigma^2}\sqrt{2}\, g_{\Sigma\pi} g_{\Lambda\pi}\,. \end{cases} \tag{8''}$$

The resulting dynamical singularities consist of cuts along the following portions of the real axis

$$-\infty \leqslant s \leqslant 0\,, \qquad 5.28 \leqslant s \leqslant 4.87\,. \tag{21''}$$

Where the shorter cut is approximated by a pole the residue at the pole is given by

$$\begin{aligned} R = -\int_{4.87}^{5.28} \mathrm{d}s\, \frac{g_{\Sigma\pi} g_{\Lambda\pi}}{\sqrt{2}b} \Big\{ \frac{W - \Lambda}{16\pi W^2} \big\{ [(W + \Lambda)^2 - \mu^2][(W + \Sigma)^2 - \mu^2] \big\}^{\frac{1}{2}} - \\ - \frac{a}{b}\frac{W + \Lambda}{16\pi W^2} \big\{ [(W - \Lambda)^2 - \mu^2][(W - \Sigma)^2 - \mu^2] \big\}^{\frac{1}{2}} \Big\}\,, \\ = \sqrt{2}\, g_{\Sigma\pi} g_{\Lambda\pi} (-.0018)\,, \end{aligned} \tag{23''}$$

where in (23″)

$$a = (\Sigma^2 - \Lambda^2)/2 - \mu^2 + s/2 - (\Sigma^2 - \mu^2)(\Lambda^2 - \mu^2)/2s\,,$$

$$b = \big\{ [s - (\Sigma + \mu)^2][s - (\Sigma - \mu)^2][s - \Lambda + \mu)^2][s - (\Lambda - \mu)^2] \big\}^{\frac{1}{2}}/2s\,.$$

The u-cut $(\Lambda + \mu)^2 \leqslant u \leqslant \infty$) in A^1, B^1 gives the following cut in the s-state amplitude:

$$-\infty \leqslant s \leqslant 4.6\,.$$

The t-cut in A^1, B^1 $(4\mu^2 \leqslant t \leqslant \infty)$ gives the following cuts in the s-state amplitude:

$$-\infty \leqslant s \leqslant 0\,, \qquad 4.68 \leqslant s \leqslant 6.15\,,$$

together with a region of possible non-continuability as in the other inelastic amplitudes. However the situation is in fact nearer to that for the elastic amplitudes. This is because the bounding curve of the region can be represented (to any physically desirable degree of approximation) by a circular cut

$$|s| = 5.35\,, \qquad (\sim \sqrt{(\Lambda^2 - \mu^2)(\Sigma^2 - \mu^2)})\,.$$

4. – Method of solution.

BJORKEN [11] has proposed a method of solving the dispersion relations for coupled scattering amplitudes for which the dynamical singularities are known. It is essentially an extension of the N/D method to coupled amplitudes and, as shown below, can be cast explicitly into that form. (Independently of BJORKEN, the method has also been given by NAUENBERG [12]).

If we have a scattering process which (for a given angular momentum isospin and strangeness) can proceed through many two body channels then the scattering amplitudes can be written as a matrix, $g(s)$. The element $g_{ij}(s)$ of this matrix is the transition amplitude from the channel i to the channel j (for a given angular momentum, isospin and strangeness). We write

$$g(s) = D^{-1}(s)\,N(s)\,,$$

where both D and N are matrices. N is adjusted so as to contain all the dynamical singularities (taken as given) and D all the (unknown) kinematical singularities.

In evaluation we are going to approximate the dynamical singularities, which largely consist of cuts, by a series of poles [1]. This makes no difference in principle to the method and for simplicity of exposition we shall adopt it henceforth. The notation taken is that s_{ij}^n are the positions of the poles in the amplitudes g_{ij}, the integer n characterizing the different poles in g_{ij}; a_{ij}^n are the corresponding residues at these poles. Then if we put

$$N_{ij}(s) = \sum_{k,n} D_{ik}(s_{kj}^n)\,\frac{a_{kj}^n}{s - s_{kj}^n}\,, \tag{52}$$

[11] J. D. BJORKEN: *Phys. Rev. Lett.*, **4**, 473 (1960).
[12] M. NAURNBERG: *Cornell University Thesis* (unpublished).

then g is identically assured of its poles with the correct residues and D may be taken to have only the kinematical singularities of g.

In the case we are considering, of $\bar{K}\mathcal{N}$ scattering and absorption, $g_{ij}(s)$ has a kinematical cut on the real axis

$$\sigma \leqslant s \leqslant \infty \,, \tag{53}$$

where

$$(\Lambda + \mu^2) \leqslant \sigma \,.$$

Consequently $D(s)$ has such singularities only.

From the Riemann-Schwarz reflection principle

$$g^*(s) = g(s^*) \tag{54}$$

so that, if we ensure that in our approximation of cuts by poles, any pole is accompanied by its conjugate, then (52) and (54) ensure that we may take

$$D^*(s) = D(s^*) \,. \tag{55}$$

Consequently we may write the subtracted dispersion relation for $D(s)$ as

$$D_{ij}(s) = D_{ij}(s_0) + \frac{s - s_0}{\pi} \int_\sigma^\infty \mathrm{d}s' \, \frac{\operatorname{Im} D_{ij}(s')}{(s' - s)(s' - s_0)} \,, \tag{56}$$

where

$$s_0 < (\Lambda + \mu)^2 \,.$$

As in the one channel case we are free to make a choice (arbitrary except for compatibility with the existence of D^{-1}) of $D(s_0)$. We choose

$$D_{ij}(s_0) = \delta_{ij} \,. \tag{57}$$

Also, for s real,

$$\operatorname{Im} D(s) = N(s) \operatorname{Im} g^{-1} \,.$$

Using eq. (6)

$$\operatorname{Im} D_{ij}(s) = - k_j \theta_j N_{ij}(s) \,. \tag{58}$$

Substituting (57) and (58) into (56) and (52) we find

$$D_{ij}(s) = \delta_{ij} - \frac{s - s_0}{\pi} \sum_{k,n} \int^\infty \frac{\mathrm{d}s' k_j \theta_j D_{ik}(s^n_{kj}) a^n_{kj}}{(s' - s_0)(s' - s^n_{kj})(s' - s)} \,. \tag{59}$$

On putting $s = s^n_{ij}$ there results a set of simultaneous equations for determining the $D_{ik}(s^n_{kj})$ from which $D(s)$, $N(s)$ and thus $g(s)$ may be found.

* * *

The author would like to acknowledge the hospitality of CERN, where this work was done, and to thank the Carnegie Trust for a travelling fellowship.

RIASSUNTO (*)

Con l'uso delle relazioni di dispersione doppia, abbiamo trovato le singolarità delle ampiezze d'onda parziali per le transizioni fra i canali accoppiati $\overline{\mathrm{K}}+\mathcal{N}$, $\pi+\Sigma$ e $\pi+\Lambda$. Si espone, in accordo con le proposte di Bjorken, un formalismo per risolvere le relazioni di dispersione, usando la condizione di unitarietà.

(*) *Traduzione a cura della Redazione.*

Photons from the Radiative Pion Decay.

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(ricevuto il 23 Dicembre 1960)

Summary. — The detection of photons from the radiative pion decay mode $\pi \to \mu + \nu + \gamma$ is reported. While there are several uncertainties in the final figures, the results are consistent with the V-A theory of weak interactions and with nuclear emulsion investigations of anomalous π-μ decays.

1. – Introduction.

The pion decay mode

$$\pi \to \mu + \nu + \gamma .$$

has been investigated theoretically by several authors [1,2]. Experimental evidence for the existence of this decay mode was first put forward by Fry [3]. More recently a similar, though more detailed, analysis was reported by Castagnoli and Muchnick [4].

Both the experiments reported involved the measurement in nuclear emulsions of the μ-meson range from π-mesons decaying at rest. For the normal two-particle decay, the μ-meson is emitted with a unique energy of 4.2 MeV. In the three-particle decay, the meson energy will in general be less. Such events can therefore be detected by looking for μ-mesons of short range. The results of Castagnoli and Muchnik [4] showed that the branching ratio for

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(1) S. Oneda and S. Kamefuchi: *Nucl. Phys.*, **6**, 114 (1958).
(2) K. Huang and F. E. Low: *Phys. Rev.*, **109**, 1400 (1958).
(3) W. Fry: *Phys. Rev.*, **83**, 1268 (1951).
(4) C. Castagnoli and M. Muchnik: *Phys. Rev.*, **112**, 1779 (1958).

production of mesons having energies between 1.85 and 3.35 MeV, relative to the normal decay, was $(1.24 \pm 0.25) \cdot 10^{-4}$.

Our interest in the decay mode was stimulated by the work of ONEDA and KAMEFUCHI [1]. These authors, and also HUANG and LOW [2], suggested that the study of the photon spectrum could yield information on the pion decay mechanism. If this proceeds via baryon-antibaryon pairs followed by a four-fermion weak interaction, then in addition to photons arising from inner bremsstrahlung, photons can also be produced in the intermediate heavy particle state. The interesting result here lies in the prediction that the two photon energy spectra should be very different, and that, while the inner bremsstrahlung spectrum is approximately of the form $1/k$, where k is the photon energy, the intermediate state photon spectrum, with some reasonable assumptions, should have a maximum somewhere between 10 and 20 MeV. The maximum photon energy possible is 29.8 MeV. Thus in principle a negative result in a search for the higher energy photons could be of great interest. Recent calculations, for example by PRAKASH [5], who made a lowest order perturbation theory calculation assuming a nucleon-antinucleon pair, a cut-off at one nucleon mass and the V-A four-fermion interaction, have shown that only the inner bremsstrahlung term should be of practical significance. Nevertheless, we continued our experimental investigation of the γ-rays emitted, and report this work here.

2. – Experimental arrangement.

The experiment was carried out using a scintillation counter arrangement (Fig. 1). 98 MeV π^+ mesons from the Liverpool Synchrocyclotron were moderated by 24 cm of polythene, traversed counter B and stopped in the $7\frac{1}{2}$ cm cube of counter S. A typical beam intensity was around 700 mesons s^{-1} stopping in S. Counter B completely covered the 5 cm diameter aperture in the shield, so that all emerging particles should have crossed B. We sought to detect γ-rays emerging from counter S for a time up to $8 \cdot 10^{-8}$ s after the arrival of a π-meson in S.

The γ-ray detector, referred to as counter G, consisted of three identical interleaved counters G_1, G_2, G_3 in coincidence with each other. Each of the three elements consisted of two sheets of plastic scintillator 1 cm thick, spaced by 3 cm and « viewed » by a single photomultiplier. On top of each sheet of plastic there was an aluminium sheet 0.122 cm thick (not shown in the figure).

Between counters S and G there was a counter, A, 2 cm thick with a sen-

[5] P. PRAKASH: *Notas de Fisica*, **4**, 16 (1958); also private communication (1959).

sitive area larger than that of G, used to give an anticoincidence signal to reject charged particles reaching G.

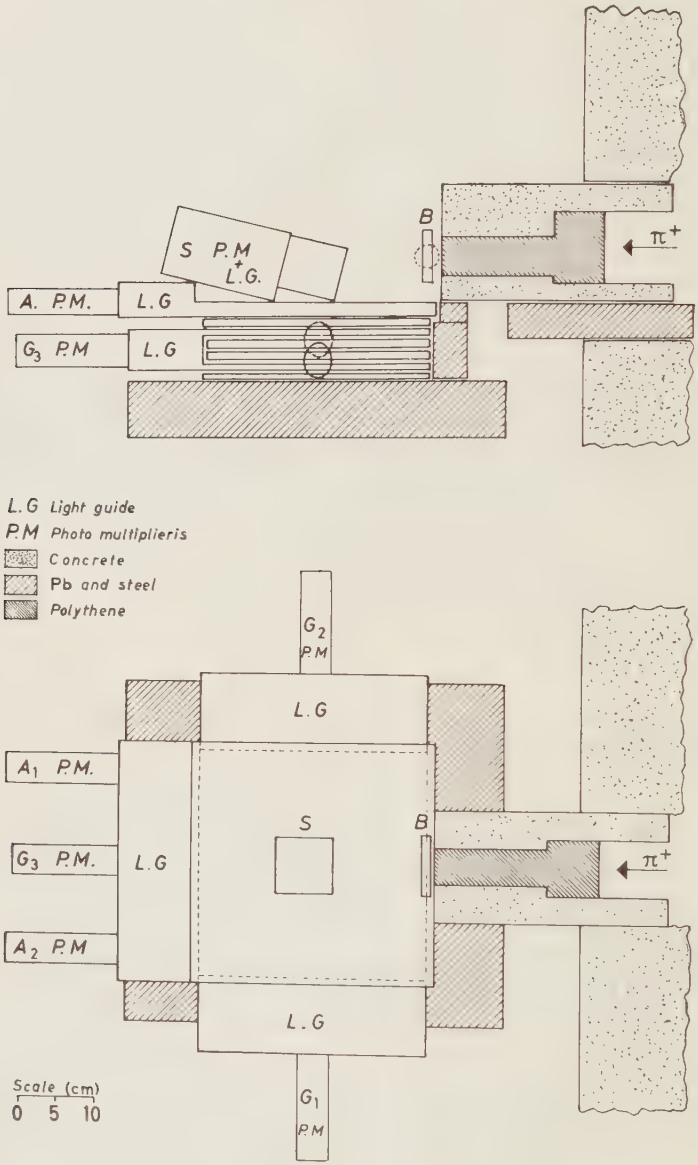

Fig. 1. – Elevation (upper diagram) and plan (lower diagram) of the arrangement of counters, shielding and beam.

A radiative decay event should have given the following signals:

i) a pulse in counter B of well-defined amplitude coincident in time with a pulse in S from a single π-meson entering through B and stopping in S;

ii) a pulse in S from the μ-meson emitted by the π-meson on decaying;

iii) no signal in A, and a signal in G coincident in time with the μ-meson pulse in S, from the γ-ray interacting in G.

Two signals were taken from counter B, a 20 nS pulse used as a coincidence signal, and an exponentially decaying pulse with a decay time of 5 μs with amplitude proportional to the signal.

The amplification of pulses from counter G was adjusted so that the counter would detect efficiently events corresponding to the traversal of one sheet of each counter by a minimum ionizing particle.

The efficiency of counter G was estimated using known Compton and pair production cross-sections (Fig. 7*a*). Including the solid angle subtended at S, it was about 1.7% for the highest energy photons, falling to less than 0.15% for photon energies less than 7 MeV.

A coincidence was registered when a coincidence $B+S$ was followed within $8\cdot10^{-8}$ s by a coincidence $G_1+G_2+G_3-A$ (referred to as $G-A$).

On the occurrence of such a coincidence, the pulses in the various counters were recorded on two oscillographs, one having a time base of duration 10 μs, the other a duration of .25 μs. Fig. 2 shows examples of the two traces. The slow trace shows the pulse from B as a step decaying exponentially, followed

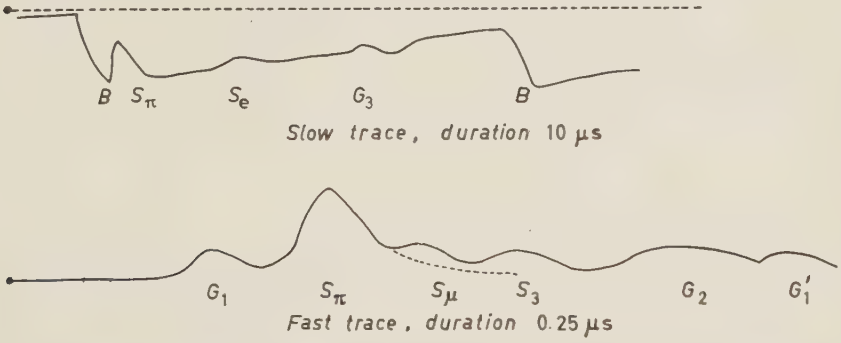

Fig. 2. – Sketches of typical oscillograph traces. Slow trace (above) of duration 10 μs. Fast trace (below) of duration 0.25 μs.

by lengthened pulses from S and G_3. A second pulse from S referred to as S_e, due to the positron from a μ-decay, and later a second pulse from B due to the next pion coming in, can also be seen. The dotted line is the deduced asymptote to the exponential. The fast trace shows G_1, followed by the pulses from S. In the example shown, these latter pulses are due to three particles: for example, S_π is a pion entering; S_μ is the muon from the pion decay; and, S_3 the positron from the muon decay. Finally we have G_2 fol-

lowed by a reflection of G_1. The reflection was introduced to calibrate the oscillograph sweep speed.

3. – Experimental results.

Two separate machine runs have been carried out. The first showed some evidence for the radiative decay, but this was not conclusive. The apparatus was then modified, the main modification being in the use of a travelling wave oscilloscope for the fast display, and a second run completed. As a consequence of the modifications made, pulse amplitudes, timing accuracies and other parameters are not easily comparable. We therefore present in detail the analysis for the second run and combine the two only at the end of the analysis. However, except for the changes in parameters, the two analyses are similar. Table I gives some of the rates measured in these runs.

TABLE I.

Run	Rate $B+S$ average/s	Time (s)	« acceptable » π's stopping	Total events	No. or «obvious» annihilation events	Estimated No. of unidentified annihilation events	No. of «G casual» events	Total no. of radiative decays	Time interval accepted $\cdot 10^{-9}$ s
Run 1	670	$1.35 \cdot 10^5$	$1.36 \cdot 10^7$	28	8	1.0	9	10.0	± 3.4
Run 2	800	$1.49 \cdot 10^5$	$1.95 \cdot 10^7$	23	6	0.8	5.7	10.5	± 1.6
			Combined data	51	14	1.8	14.7	20.5	

A total time of 42 hours was spent during the second run, recording $B+S+G-A$ events, with a total of 5200 pairs of photographs of oscillograph traces. The beam intensity was chosen to make a compromise between the rate of genuine events and backgrounds. We monitored, at approximately hourly intervals, the rates of:

i) $B+S$ events, as a measure of the π^+ flux;

ii) $G-A$ events, as a check on background processes;

iii) $B+S+G$ events, due to positrons from μ's decaying within $8 \cdot 10^{-8}$ s of the π.

We photographed $B+S$ events at a very low beam intensity to give a check on the efficiency of seeing the e^+ on the slow trace and the muon on the fast; also to give the μ-meson pulse height distribution for the 4.2 MeV μ-mesons from the normal π-decay. The accuracy of measuring the time of the

S_μ pulse relative to G_1 was checked by using photos of $B+S+G$ events. Cosmic ray photos of fast μ-meson events $S+G$ at full gain and half gain for the G counters gave a calibration of G counter pulse heights.

4. – Analysis of data.

We demanded the following conditions to be satisfied before an event was accepted for further analysis. These selection criteria reduced our efficiency for detecting radiative decay events by the factors given in brackets.

a) The slow trace had to show a B signal of the amplitude expected for a single π-meson traversing the counter, and we required that the trace start with little or no displacement from the base line, determined from the asymptote of the exponentially decaying B pulse (0.86). These criteria select events where only one π-meson enters the counter at the time of interest, and none has entered during the previous 5 μs.

b) The slow trace had to show a pulse attributable to a positron decay in S (0.9).

c) The slow trace had not to show a second π-meson entering, that is a second B pulse, before the positron pulse (0.9).

d) The fast trace had to show a pulse attributable to a stopping π-meson (0.86) emitting a μ-meson late enough for the μ amplitude to be measured (0.36.) In general a μ emitted after the peak of the S pulse was acceptable.

e) The fast trace had to show G_1 and G_2 pulses in coincidence with one another, and with the pulse attributable to the μ-meson. The fluctuations in the G-μ coincidence time were measured by photographing events where positrons from the μ-decay in S entered G. For the first run this was $\pm 2.1\ nS$, for the second run this was $\pm 1.0\ nS$. In each run we accepted only coincidences within ± 1.6 *S.D.* of the expected « zero » delay (0.89).

After imposing these criteria, and correcting for the camera dead time (0.85) and the occasional loss of photographs (0.95) we were left with 18% of the incident π-mesons being « acceptable » for giving rise to acceptable radiative decay events.

Having imposed these selection criteria we were left with 51 events. We will now discuss the possible origins of these 51 events.

One source of spurious events which we refer to as « annihilation » events, simulating radiative decays, is the following. On the fast trace the S_π pulse is due to a stopping π-meson, but its μ-meson is emitted very early, escaping

observation; its decay electron simulates an S_μ pulse in S, and then annihilates in flight, and the annihilation photons are detected in G. On the slow trace an apparent « positron » pulse appears but this is not related to the stopping π-meson. It should be noted that this background cannot be reduced by improving the resolving time of the fast oscillograph. The slow oscillograph was used, and selection criteria *a*), *b*) and *c*) were imposed in order to eliminate this class of events by making it improbable for a μ-meson from a second π-meson to give a positron signal on the slow trace. Without these selection criteria this class of events was 40 times more numerous than the radiative decays.

An examination of an unbiased sample of « annihilation » events for which the selection criteria *b*) of a positron pulse was relaxed, showed that only 1 in 8 of these could be confused with any radiative decay, since either the real μ pulse as well as the positron simulating the μ was present, or the apparent μ pulse was too large.

Examination of the 51 selected events enabled 14 to be identified and rejected as « obvious » annihilation events. This leaves 37 events, of which probably 2 are due to unidentified « annihilation » events.

A second source of spurious radiative decay events can arise from events where the π and e pulses on the slow and fast traces are genuinely due to such particles but the μ-meson from this π decays too early to be identified. If, then, the S_μ and « G » pulses are in fact unrelated to the π-meson, but have a common origin and are therefore si-

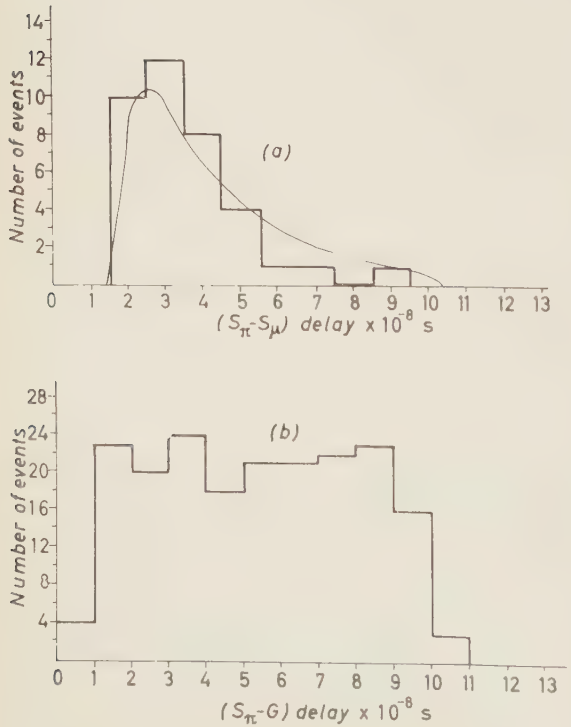

Fig. 3. – *a*) Frequency distribution for the μ decay relative to the π for the 37 time-coincident events. The histogram shows the measured distribution. The continuous curve is that predicted using a π lifetime of $2.6 \cdot 10^{-8}$ s. *b*) Delay distribution of casual G pulses with respect to π-mesons, showing no significant modulation with the synchrocyclotron r.f. period of $6 \cdot 10^{-8}$ s.

multaneous, we have a class of events, « *SG* casuals », that can give rise to a spurious background. Here again improvement in the fast resolving time would not reduce the background.

Evidence that such events, if any, are rare, is seen in Fig. 3*a*), which shows the frequency distribution of $S_\pi - S_\mu$ delay times for the remaining 37 events. For radiative decay events this should show an exponential decay with the π-meson lifetime. For « *SG* casuals » the distribution should follow any time modulation of the background with r.f. accelerating frequency (the time for one cyclotron orbit of a 380 MeV proton is about $6 \cdot 10^{-8}$ s). We found no time modulation of the background (Fig. 3*b*), and therefore conclude from Fig. 3*a* that the « *SG* casual » rate probably contributes no counts, and cannot be contributing more than 2 or 3 counts to the 37 accepted events.

It may be remarked that the timing distribution of Fig. 3*a* confirms the conclusion previously reached that there are few « annihilation » events amongst the 37 accepted events, since these also would be uniformly distributed over this time interval.

A third source of spurious radiative decays arises from events where the S_π and S_μ pulses on the fast trace, and the S_e pulse on the slow trace are all genuinely due to such particles, but the *G* pulse is unrelated to these particles. Such « *G* casual » events contributed the greater part of the events recorded.

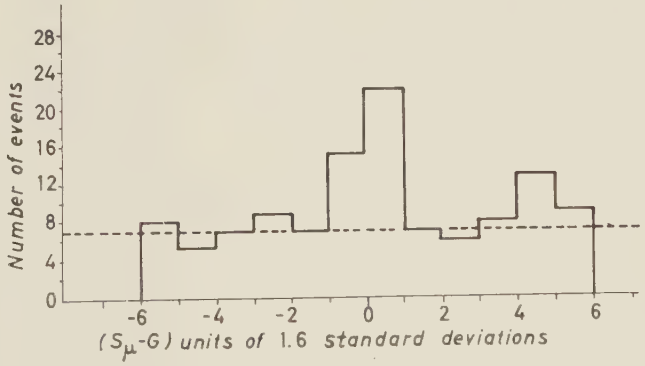

Fig. 4. – Frequency distribution of the *G* pulse delay relative to the μ, showing a contribution above the « *G* casual » background at zero delay. The two machine runs are combined by plotting the delay in terms of the standard deviations for the timing measurementes (Table I).

In spite of attempts to shield the *G* counters to reduce this rate there were 7 (*G*-*A*) counts s^{-1}. Fig. 4 shows the frequency distribution of events satisfying all the selection criteria except *e*), and excluding « obvious » annihilation events, as a function of the delay between the S_μ and *G* pulses on the fast trace.

From this it is seen that there is a contribution of events at zero delay above the background. The selection criterion *e*) was imposed to select radiative decays against this background, and the travelling wave oscillograph was built to improve the resolving time and reduce this background. From Fig. 4 it may be calculated that 15 « *G* casual » counts are expected in the time interval over which our 37 events are accepted.

Thus we have found 37 events in all, of which 2 are expected to be due to « annihilation » events and 15 to « *G* casual » events. We attribute the remaining 20 events to genuine radiative decays.

We investigated the distribution of S_μ amplitudes and the combined *G* amplitudes. Fig. 5*a* shows S_μ amplitudes from « *G* casual » events. These should represent an unbiassed sample of μ-mesons of the maximum energy. The distributions from both runs have been combined by scaling the first run to give the same mean as the second. Fig. 5*b* shows the amplitude distribution for the 37 time coincident events. We do not expect to resolve S_μ pulses of amplitude less than 0.1, but should be at least 50 % efficient at finding pulses of 0.15, with almost 100 % efficiency at 0.2. This gives direct evidence for μ-mesons with energy below the maximum value.

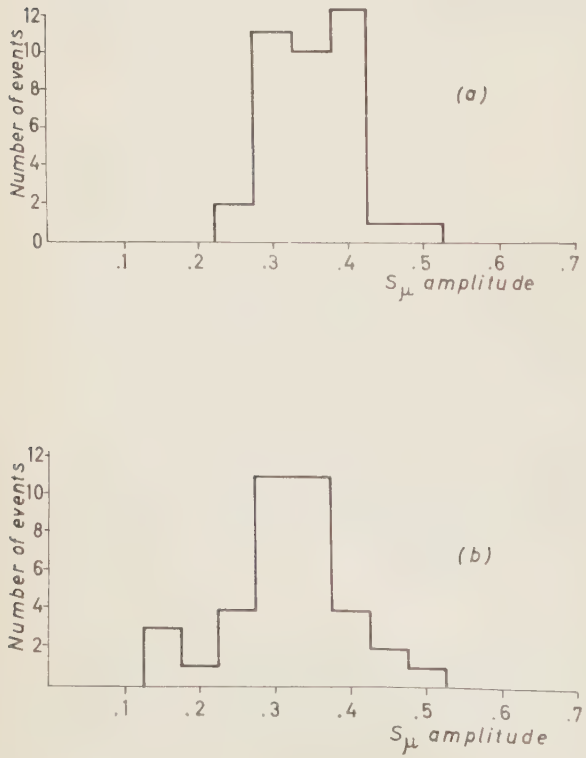

Fig. 5. – *a*) Distribution of S_μ amplitudes for normal $\pi \to \mu + \nu$ decays. *b*) S_μ amplitudes for the 37 time-coincident events, showing evidence for some μ-mesons of reduced energy.

The *G* amplitude distributions are shown in Fig. 6. Fig. 6*a* shows the distribution for minimum ionizing particles traversing one sheet of each of the *G* counters; Fig. 6*b* is the distribution for the « *G* casual » events, and Fig. 6*c* is the distribution for the 37 time coincident events. We note, firstly, that for a photon to be detected it must produce a total ionization at least comparable with a minimum

ionizing particle in three sheets (about 6 MeV); secondly, that there is little difference between Figs. 6*b* and 6*c*; thirdly, that there is no special peaking of

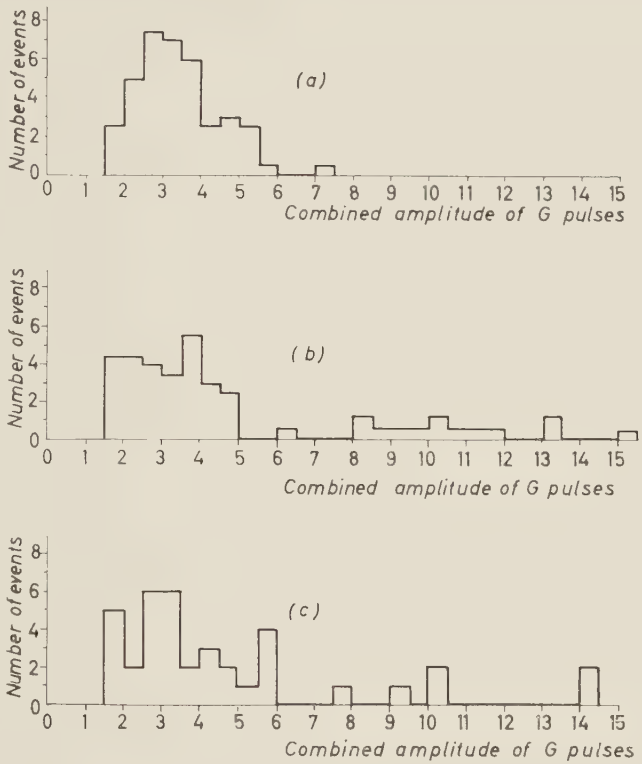

Fig. 6. – Combined pulse amplitudes from the three G counters; *a*) minimum ionizing particles traversing one sheet of each counter; *b*) « G casual » events; *c*) the 37 time-coincident events.

pulses just above the threshold. In fact, although the distribution is compatible with the predicted radiative decay spectrum it would, for instance, be consistent with all the time coincident events being really just « G casuals ».

5. – Discussion.

For the calculation of the branching ratio we require an estimate of the γ counter efficiency. For the detection of the higher energy γ-rays it is not difficult to make this fairly accurately. Indeed, a check can be made on the expected rate of « annihilation » events, this rate being about 0.47 per minute whereas the measured rate is 0.61 ± 0.08. The estimate of the pro-

bability of detecting an « inner bremsstrahlung » event is more complicated, as the Compton electrons produced will only occasionally have sufficient energy to penetrate the three counters, $G_1\,G_2\,G_3$. Scattering of the electrons is more important, and double processes have also had to be considered. Fig. 7*a* shows

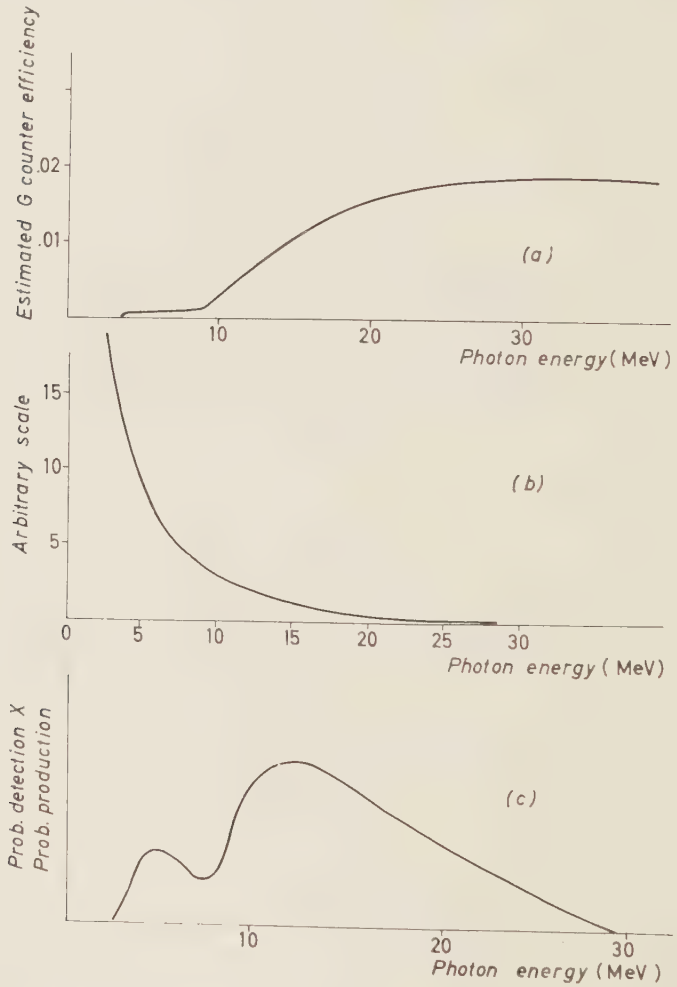

Fig. 7. – *a*) Estimated total G counter efficiency. *b*) Function I_2, proportional to the photon differential energy spectrum. *c*) The product of the two curves, showing the bias in favour of detecting the higher energy photons.

the estimated G counter efficiency; Fig. 7*b* is the function I_2 of Prakash proportional to the inner bremsstrahlung differential energy spectrum ([5]); and Fig. 7*c* the product of the two curves, showing the range of photon energies accepted. From this area we find a total probability of detecting a radiative

mode from a π in S, of $4.4 \cdot 10^{-7}$. Multiplying this by the total acceptable number of π's stopping in S (Table I), and the probability that the photon did not interact in S or A before reaching G_1 (0.89), we have an expected number of events of 11.6. This figure must therefore be compared with the expected background of 17 and the observed total of 37 events.

6. – Conclusion.

The experiment has clearly detected γ-rays from the radiative decay mode of the π-meson. The frequency and energy spectrum of these γ-rays is consistent with the spectrum predicted by considering « inner bremsstrahlung » only. However, the statistical errors, background effects, μ-meson detection limitations, and the difficulties of estimating the efficiency of the γ-ray counter render impossible a sensitive test of the theoretical predictions.

* * *

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S. L. de S. Barros wishes to thank Manchester University for a grant received during the course of the experiment, and D. F. Falla a teaching fellowship from the Nuffield Foundation.

RIASSUNTO (*)

Si riferisce sulla rivelazione dei fotoni provenienti dal modo di decadimento radiativo dei pioni $\pi \to \mu + \nu + \gamma$. Pur essendoci molte incertezze nei valori finali, i risultati sono concordanti con la teoria V-A delle interazioni deboli e con le ricerche a mezzo di emulsioni nucleari sui decadimenti anomali π-μ.

(*) *Traduzione a cura della Redazione.*

Ionic Recombination in Liquid Helium (*).

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Istituto di Fisica dell'Università - Padova

(ricevuto il 7 Gennaio 1961)

Summary. — The volume recombination coefficient of ions in liquid helium II has been measured down to 0.8 °K by a new experimental procedure. The results can be well interpreted in terms of the Langevin and Harper theories usually valid for ordinary gases, the excitations taking now the role of the neutral molecules and the superfluid phase the place of the empty space.

Introduction.

The experimental determination of the value of the coefficient of recombination in liquid Helium is important to test the validity of the present picture of the behaviour of ions in liquid helium, as electric charges moving in a gas of thermal excitations with which the ions collide. Also interesting is the possibility of measuring the ionic recombination coefficient in a liquid, using superfluid helium, which is probably the purest substance existing, and avoiding the difficulties due to the presence of impurities; a factor which has always plagued research in the field of ionic recombination.

1. – Principle of the method.

We have measured the volume coefficient of recombination of the ions in liquid He II using a new and direct method. Essentially our method consists of sending two ionic beams of opposite sign one against the other, and measuring the loss of the charges suffered by each beam because of recom-

(*) Work supported in part by Contract DA91591 EUC 1290 with the European Research and Development, U.S. Army, and in part by the C.N.R. and I.N.F.N.

(**) Present adress: Istituto di Fisica, Università di Roma, Italy.

(***) Present adress: The Clarendon Laboratory, Oxford, England.

bination. As may be seen in Fig. 1, on the silver electrode A, which is connected to the power supply, we deposited a very homogeneous and well defined layer of ^{210}Po, on an accurately measured rectangular area. In the liquid there will be a 0.2 mm thick densely ionized layer, from which it will be possible to draw a beam of ions of one sign, by means of an electric field. From now on we will refer to this source carried by electrode A with the name of « source 1 ». The electrode A can be moved up and down, parallel to itself, running in two grooves machined into the wall of the brass cell containing the whole apparatus. In front of electrode A, and parallel to its surface, there are four more silver electrodes of similar construction. The two outer ones, labelled *G.E.* are guard electrodes used to guarantee the uniformity of the field over all the central region where the collecting electrodes B and C are fixed. These two electrodes are the same in dimensions and construction, except that one of them, namely electrode B, carries a source in every way identical to source 1. Each one of these electrodes is separately connected to a vibrating reed electrometer and the output of the electrometer goes to a recording unit, in such a way that it is possible to record simultaneously and separately ionic currents collected on each of the electrodes B and C. Obviously on electrode C a current I_1 will be collected, due to the ions extracted from source 1, when, owing to the position of electrode A this source is in front of C, and no current will be collected in all other positions of A. On electrode B on the other hand a current I_2, corresponding to the ions extracted from source 2 will be always read; and to this current will be added the current I_1 when the position of A is such that source 1 is in front of electrode B, but not directly facing source 1. But when the two sources are exactly in opposition, the total current measured on electrode B will be the sum of I_1 and I_2 minus as amount ΔI equivalent to the losses produced in the overlapping ionic beams of opposite sign by the recombination of the ions during the flight between A and B. Therefore the current measured on electrode B will be I_2, I_1+I_2, or $I_1+I_2-\Delta I$ according to the position of electrode A.

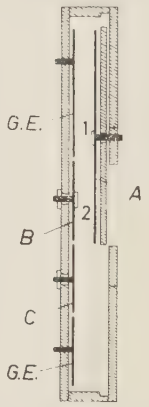

Fig. 1. – Schematic cross-section of the recombination apparatus. *G.E.* are two grounded guard electrodes, and electrode A is connected to the power supply. Electrodes B and C are each separately connected to a vibrating reed electrometer.

To correlate ΔI with the recombination coefficient, we introduce the current density $J = I/S$, S beeing the area of the source, and write for the positive ion

$$I_+ = n_+ e\mu_+ E - \Delta n_+ e\mu_+ E \,, \tag{1}$$

μ_+ beeing the mobility and n_+ the average number density of positive ions, and Δn_+ its decrease due to recombination in the path between the electrodes, which are at distance l. We assume $\Delta n_+ \ll n_+$. Then by definition

$$\Delta n_+ = n_+ n_- \alpha \Delta t = n_+ n_- \alpha \frac{l}{\mu_+ E}, \tag{2}$$

and therefore

$$J_+ = J_+^0 - e n_+ n_- \alpha l, \tag{3}$$

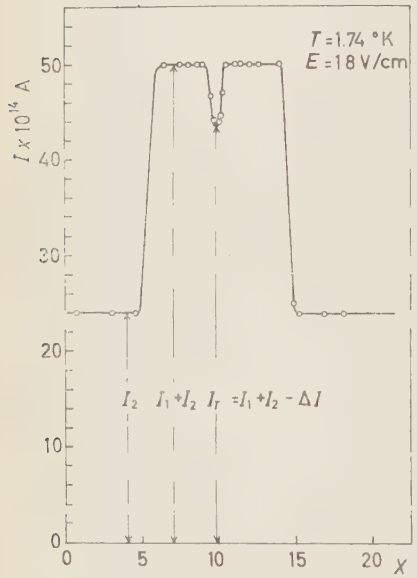

Fig. 2. – Ionic current intensities measured on electrode B in the various positions of electrode A; these positions are reported in arbitrary units on the X axis.

where J_+^0 stands for the current density of the positive beam with no recombination losses. A similar expression will hold for the negative ions as well,

$$J_- = J_-^0 - e n_+ n_- \alpha l \tag{3'}$$

and therefore

$$\Delta J = 2 e n_+ n_- \alpha l, \tag{4}$$

from which

$$\alpha = \frac{1}{2l} \frac{\Delta J}{J_+^0 J_-^0} \mu_+ \mu_- e E^2. \tag{5}$$

The measure of ΔI and of currents I_1 and I_2 is therefore sufficient, at each field intensity and temperature, to obtain the ionic recombination coefficient, by means of the expression (5). Fig. 2 shows the currents collected on electrode B during a complete run of electrode A. It is evident that these data are by themselves sufficient to determine α. The value of I_1, separately registered on electrode C was used only to check the conditions of operation of the apparatus.

2. – Evaluation of the correction factors.

In practice one is obliged to account for various departures from the ideal behaviour outlined above. Indeed there are, first of all, space charge effects which tend to reduce the ionic currents I_1 and I_2, when the two beams do not overlap. When on the contrary they do overlap, the space charge effects

cancel out, the ions of the beams being of opposite sign and almost equal in number density. The resulting effect is that the ΔI directly read is less than the true one due to recombination.

As shown in Fig. 3, if the applied field intensity is E_0 and E^* is the value of the space charge field obtained with eq. (4) for the corresponding current density, then the current actually measured is smaller than the current one would have obtained in the absence of space charge effects, by the amount I^*. When the two ionic beams overlap, the current would therefore increase (instead of decreasing) if there were not recombination. The observed ΔI is therefore the net result of the decrease of current due to recombination, and of the increase discussed above. The true ΔI is then the observed one plus I_+^* and I_-^*.

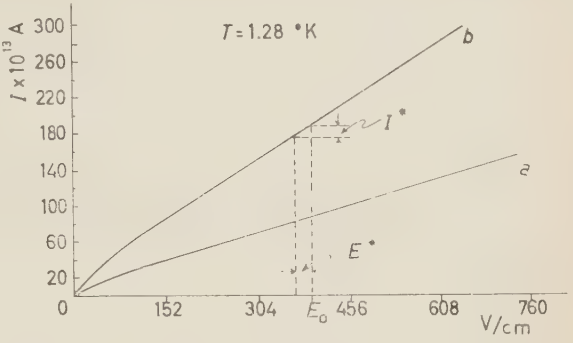

Fig. 3. – Ionic current intensities measured on electrode B at increasing field intensities. The curve a is obtained when A is in such a position that only the current I_2 is collected on B. The curve b is the sum of currents I_1 and I_2 collected together on electrode B, but with no recombination loss. I^* is also indicated (see text).

Another group of corrections originate from the fact that the cross-sections of the two ionic beams are larger than the geometric area of the sources. This is because of the ionization produced by the α-particles shot out side-ways beyond the edge of the radio-active area, and because of the lateral spreading of the ionic beams caused by space charge and diffusion. The enlargement of the source produced by the beyond-the-edge ionization is easily calculated since the range of the α-particles in liquid He is well known.

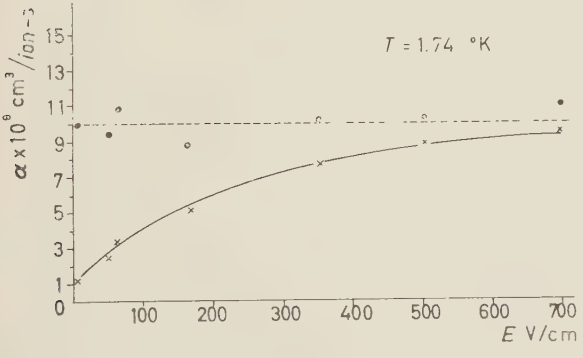

Fig. 4. – Experimental values of the volume recombination coefficient α obtained at different field intensities. The crosses are the values calculated from relation (3) without the corrections for space charge and spreading of the beams. The final, corrected, values are the circles.

The effects of diffusion and space charge produce only moderate spreading of the beams in our conditions. The calculation of these effects is easily accomplished since the ionic mobilities are known, and the diffusion coefficient can be deduced from the mobility by means of Einstein's relations.

In any event our apparatus gives us also the possibility of a direct evaluation of the overall enlargement of the beams, comparing the experimental with the ideal corresponding figure one should obtain on the basis of the geometrical construction of the cell.

All the above specified effects, except the enlargement of the source due to over-the-edge ionization are affected by temperature and by the intensity of the applied electric field E. In Fig. 4 we give the results of seven separate runs with different field intensities, all at the same temperature of 1.74 K. It is to be noticed that the final values of α do not depend on the field intensity.

3. – Experimental results.

In Fig. 5 and in Table I we give a resumé of all the data obtained for the ionic recombination coefficient; these data in Fig. 5 are plotted *vs.* the

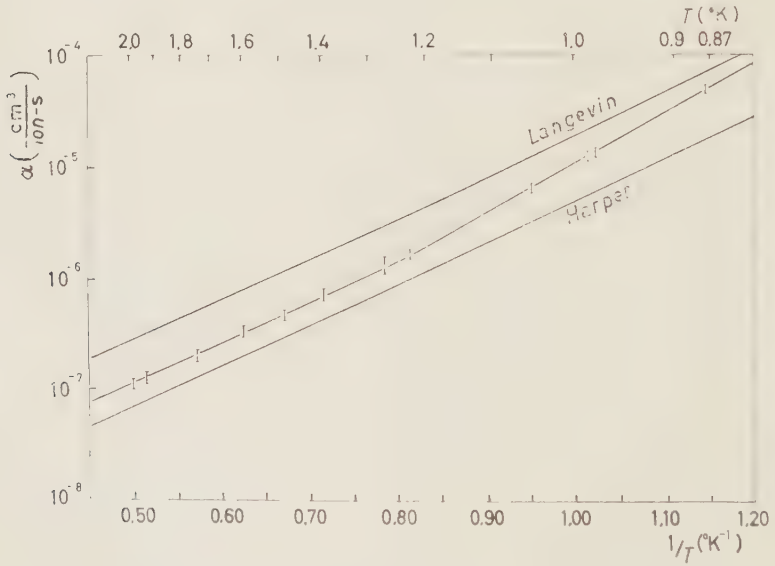

Fig. 5. – Volume recombination coefficient of ions in liquid He *vs.* the inverse of absolute temperature. The upper and the lower lines give the values calculated from Langevin's and Harper's theory respectively. The central curve is the best fit of our experimental results.

inverse of absolute temperature. The general behaviour is an exponential increase of α with the reciprocal of absolute temperature, which is like the

dependence of the ionic mobility on temperature in liquid He II. This increase can be represented by the expression:

$$\alpha = A \exp[\Delta/kT], \tag{7}$$

with $\Delta/k = 8.7$ °K, and k is the Boltzmann constant.

TABLE I. – *The experimental values of the recombination coefficient* α [cm³/ions s] *at different temperatures* T [°K] *and applied fields* E [V/cm].

T	E	α	T	E	α
2.0	66.6	$1.1 \cdot 10^{-7}$		166.6	$8.01 \cdot 10^{-7}$
				90.9	$7.23 \cdot 10^{-7}$
	380.0	$1.3 \cdot 10^{-7}$	1.40	18.9	$6.83 \cdot 10^{-7}$
	199.0	$1.09 \cdot 10^{-7}$		9.3	$7.81 \cdot 10^{-7}$
	90.9	$1.27 \cdot 10^{-7}$		6.6	$6.44 \cdot 10^{-7}$
1.94	49.5	$1.05 \cdot 10^{-7}$			
	18.3	$1.11 \cdot 10^{-7}$		190.0	$1.37 \cdot 10^{-6}$
	15.0	$1.39 \cdot 10^{-7}$		90.9	$1.53 \cdot 10^{-6}$
	9.3	$1.25 \cdot 10^{-7}$		63.5	$1.12 \cdot 10^{-6}$
			1.28	42.0	$1.20 \cdot 10^{-6}$
	705.0	$2.2 \cdot 10^{-7}$		18.9	$1.41 \cdot 10^{-6}$
	500.0	$2.04 \cdot 10^{-7}$		6.6	$1.25 \cdot 10^{-6}$
1.74	350.0	$2.01 \cdot 10^{-7}$			
	166.6	$1.78 \cdot 10^{-7}$		49.0	$1.51 \cdot 10^{-6}$
	63.5	$2.16 \cdot 10^{-7}$	1.23	30.3	$1.43 \cdot 10^{-6}$
	50.0	$1.88 \cdot 10^{-7}$		9.3	$1.72 \cdot 10^{-6}$
	6.6	$1.96 \cdot 10^{-7}$			
				29.9	$6.91 \cdot 10^{-6}$
			1.05	12.3	$5.90 \cdot 10^{-6}$
				9.3	$6.33 \cdot 10^{-6}$
	66.6	$3.07 \cdot 10^{-7}$			
1.6	18.9	$3.60 \cdot 10^{-7}$		49.0	$1.23 \cdot 10^{+5}$
	6.6	$3.22 \cdot 10^{-7}$	0.98	30.3	$1.30 \cdot 10^{-5}$
				18.9	$1.40 \cdot 10^{-5}$
				9.3	$1.42 \cdot 10^{-5}$
	180.0	$4.51 \cdot 10^{-7}$			
1.49	166.6	$4.51 \cdot 10^{-7}$	0.87	19.0	$4.51 \cdot 10^{-5}$
	18.9	$4.76 \cdot 10^{-7}$		9.3	$5.01 \cdot 10^{+5}$

This is what one gets if one tries to fit all the data with a single straight line. But it is evident that a better fit of the data can be obtained by using two straight lines of slightly different slopes at high and at low temperatures

(actually between 2 °K and 1.3 °K with Δ/k of 8.3 °K and at temperatures lower that 1.3 °K with a second slope of 9.9 °K).

Looking closely at the mobility data obtained by MAYER and REIF [1] it can be seen that also the experimental points of the mobility obtained by these authors between 2 K and 1.3 °K fall on a line with a slope smaller than the points between 1.3 °K and 0.8 °K. That it should be so is readily explainable in terms of the decrease of the Landau energy gap, so well shown by the neutron diffraction experiments, and usually accounted for in terms of non ideality of the « roton gas ». Anyway the increase of steepness in the mobility plot is very small and insufficient to explain the observed increase of α.

Let us now try to explain our results in the frame of our general picture of the behaviour of ions in superfluid helium.

4. – Discussion of the results.

To understand the meaning of our results we must notice that the densities in the ionic beams of our apparatus are of the order of 10^7 ions/cm^3. The average interionic distances are therefore 2 to 3 μm. The average distance d_0 at which the attractive electrostatic forces between ions of opposite sign prevail over the effects of thermal agitation, so that the ions are dragged closer and finally recombine, can be calculated from the relation: $d_0 = 2e^2/3kT$ suggested by THOMSON, and give a value four to seven times smaller than this average interionic distance. Therefore most of the movement of the ions takes place at a distance greater than d_0, and consists of a disordered thermal motion with a unidirectional drift in the external field. During their movement the ions collide with the thermal excitations exchanging with them kinetic energy and momentum, but do not collide with the superfluid phase [2]. When, by accident, an ion in its wandering comes nearer than d_0 to an ion of opposite sign, the ions start drifting one towards the other, until the charges recombine. A process like the one outlined above is met in the volume ionic recombination in moderately compressed gases, to which Harper's theory applies [3], the only difference being that the « gas of excitations » takes now the place of the gas of neutral molecules, and the superfluid phase takes the role of the empty space between the gas molecules. Harper's theory gives for the recombination

[1] L. MAYER and R. REIF: *Phys. Rev.*, **110**, 279 (1958).

[2] See the review article by G. CARERI in *Progress in Low Temperature Physics*, vol. III (Amsterdam, in the press).

[3] W. R. HARPER: *Proc. Camb. Phil. Soc.*, **28**, 219 (1932); *Phil. Mag.*, **18**, 97 (1934); **20**, 740 (1935).

coefficient the value:

$$\alpha_{\mathrm{H}} = \pi e \varepsilon (\mu_+ + \mu_-) \,, \tag{8}$$

where ε has generally a value near the unity. Harper's result is rather close, at it is well known, to the Langevin's one which is the preferential recombination coefficient at very high pressures ([4]). Langevin's result is:

$$\alpha_{\mathrm{L}} = 4\pi e (\mu_+ + \mu_-) \,. \tag{9}$$

In Fig. 5 we have drawn, in addition to our experimental results, two continuous lines. The lower curve gives Harper's α calculated from the experimental mobility values of Mayer and Reif taking $\varepsilon = 1$, and the upper curve is calculated from Langevin's expression. Our high temperature data in the interval 2 °K ÷ 1.3 °K fall on a line parallel to Harper's but are higher in value, what would lead to a value of ε somehow larger then unity. The low-temperature part of the experimental curve instead approaches the value obtained using Langevin's formula. Since the « Thomson distance » d_0 increases as T decreases, this possibly means that at temperatures lower than 1.3 °K, d_0 becomes comparable with the mean interionic distance in the ionic beams, so that the ionic plasma approaches the situation assumed by Langevin in his theory. In other words it turns out that the Harper's and Langevin's expressions are the two limiting laws for respectively large and small values of the average interionic distances with respect to d_0.

The good agreement with the Harper and Langevin theories which are valid in gases for ion-ion and not for electron-ion recombination, confirms once more the massive model of the negative ion already suggested by us in a previous paper ([5]).

5. – Concluding remarks.

We want to emphasize the extremely small scatter of the experimental data if compared with the best recombination measurements obtained in gases. This fact is presumably due to the very high purity of liquid He II, in which the only extraneous substances can be the isotope of mass 3, and even this impurity, amounting to less than one part in a million, should not affect our results at temperatures higher than about 0.5 °K.

([4]) For a general revue of the subject of recombination in gases, and in particular for Harper's and Langevin's theories, see for instance L. Loeb: *Basic Processes of Gaseous Electronics* (Berkeley, 1955).

([5]) G. Careri, U. Fasoli and F. S. Gaeta: *Nuovo Cimento*, **15**, 774 (1960).

Also noterworthy is the fact that the absolute magnitude of the recombination factor is such that an ionic plasma as superfluid Helium does not decay very quickly, and therefore collective effects (for example plasma oscillations) might have time to originate and eventually to be observed.

Finally we note that the coefficient of recombination that we have measured is the volume coefficient of recombination, and therefore it cannot be used to calculate the recombination of the ions generated along the track of an α-particle. But when the source of ionization is a β-emitter, the ions are generated in rather widely separated pairs, and the situation in the area of the source should not differ too much from the one met in volume recombination, with the exception that most of the ionic pairs are separated by distances lesser than d_0. This situation is identical to the one called preferential recombination to which Langevin's theory applies. Therefore we think very likely that the coefficient of recombination in β-irradiated liquid helium assumes the value of eq. (9).

This method can be extended to other non-conducting liquids, if they can be made sufficiently pure.

RIASSUNTO

Il coefficiente di ricombinazione di volume di ioni in elio II è stato misurato con una nuova tecnica sperimental fino a 0.87 °K. I risultati possono essere bene interpretati in termini delle teorie di Langevin e di Harper valide per i gas ordinari, ma ora prendendo le eccitazioni al posto delle molecole neutre ed il superfluido al posto de l'e spazio vuoto.

Pion-Hyperon Scattering.

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(ricevuto il 12 Gennaio 1961)

Summary. — A dispersion theoretic technique, suggested by Feldman, Matthews and Salam, is used to derive the static equations for pion-hyperon scattering and it is shown that the inclusion of the $\bar{K}\mathcal{N}$ channel does not change the conclusions on the $J=\frac{3}{2}$, $I=1$ p-wave, π-Y resonance, given by Amati, Stanghellini and Vitale and by Capps.

1. – Introduction.

Feldman, Matthews and Salam [1] suggested that we may write down a dispersion relation for the quantity $B_{l\mp}(T_{l\pm})^{-1}$, where $T_{l\pm}$ is the amplitude for $J=l\pm\frac{1}{2}$ scattering and $B_{l\mp}$ is the corresponding Born amplitude. Now, unitarity gives $\mathrm{Im}\,(T_{l\pm})^{-1}=-k$ on the right-hand physical cut ($S_J=1+2ik^{\frac{1}{2}}T_Jk^{\frac{1}{2}}$). So, if the approximation of neglecting the left-hand cut can be made, then, we have the dispersion relation

$$B_{l\pm}(\omega)\,\mathrm{Re}\,(T_{l\pm}(\omega))^{-1}=1-\frac{\omega-\omega_B}{\pi}P\int\frac{B_{l\pm}(\omega')k'\,\mathrm{d}\omega'}{(\omega'-\omega)(\omega'-\omega_B)}\,, \tag{1}$$

where ω_B is the pole of the Born term and a subtraction has been made at this point. Using eq. (1) FMS [1] reproduced very simply the Chew-Mandelstam result on π-π scattering and the Chew-Low equation for the (3, 3) amplitude in the π-$\mathcal{N}$ scattering.

In deriving the static equations for π-Y scattering in Section **2**, we shall take into account the Σ-Λ mass difference and we shall find that $B_{l\pm}(\omega)$ breaks

[1] G. Feldman, P. T. Matthews and A. Salam: *Nuovo Cimento*, **16**, 549 (1960).

up into two or three terms, the poles of which are different. A slight generalization of (1) is then necessary. For example, if $B_{l\pm}(\omega) = B^1_{l\pm}(\omega) + B^2_{l\pm}(\omega)$ where $B^1_{l\pm}(\omega)$ has the pole at $\omega = \omega_1$, and $B^2_{l\pm}(\omega)$ at $\omega = \omega_2$, then we take the dispersion relation

$$(2) \qquad B_{l\pm}(\omega)\,\mathrm{Re}\,(T_{l\pm}(\omega))^{-1} = 1 - \frac{\omega-\omega_1}{\pi} P\int \frac{B^1_{l\pm}(\omega')k'\,\mathrm{d}\omega'}{(\omega'-\omega)(\omega'-\omega_1)} - \\ - \frac{\omega-\omega_2}{\pi} P\int \frac{B^2_{l\pm}(\omega')k'\,\mathrm{d}\omega'}{(\omega'-\omega)(\omega'-\omega_2)} .$$

In Section **3**, the position of $J=\frac{3}{2}$, $I=1$ p-wave resonance is determined when effects of $\bar{K}.N$ channel are included and it is found that qualitatively the position remains the same as that given by Amati, Stanghellini and Vitale [2]. In Appendix I, we derive the π-Y static equations exactly in the Chew-Low [3] form, reproducing very simply the results of Capps [4]. Static equations for π-Y scattering, with somewhat different approximations, have been derived by Capps and Nauenberg [5]. In Appendix II, a brief discussion on the condition for resonance and on how a closed channel can affect the position of resonance of an open channel, is given, taking the two channel case as a simple illustration.

2. – The pion-hyperon static equations.

Amati, Stanghellini and Vitale [6] have studied the low-energy pion-hyperon scattering in the static limit, using a field theoretical model which takes into account the Σ-Λ mass difference and the possible inequality of $\Sigma\pi$ and $\Lambda\pi$ coupling constants. Using our dispersion technique, we can reproduce their results very simply.

Since ASV discuss their results in terms of the K-matrix we introduce it also in our discussion. The K-matrix, in our case is given by

$$(3) \qquad K = k^{\frac{1}{2}}(C)^{-1}Bk^{\frac{1}{2}} \qquad \text{where } C \equiv B\,\mathrm{Re}\,T^{-1}.$$

Let us first consider

$$I = 0\,, \qquad J = \tfrac{3}{2}\,, \qquad l = 1\,.$$

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[2] D. Amati, A. Stanghellini and B. Vitale: *Phys. Rev. Lett.*, **5**, 524 (1960).

[3] G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu: *Phys. Rev.*, **106**, 1337 (1957); G. F. Chew and F. E. Low: *Phys. Rev.*, **101**, 1570 (1956).

[4] R. H. Capps: *Phys. Rev.*, **119**, 1753 (1960).

[5] R. H. Capps and M. Nauenberg: *Phys. Rev.*, **118**, 593 (1960).

[6] D. Amati, A. Stanghellini and B. Vitale: *Nuovo Cimento*, **13**, 1143 (1959); hereafter referred to as ASV.

In this case, the Born approximation is given by

$$B_3^0 = \frac{2}{3} k^2 \left[\frac{f_\Lambda^2}{\omega - 2\Delta} - \frac{2 f_\Sigma^2}{\omega - \Delta} \right]. \qquad \omega = W - m_\Lambda,\ \Delta = m_\Sigma - m_\Lambda,$$

The first term has the pole at $\omega = 2\Delta$ while the second term has it at $\omega = \Delta$. Following our dispersion representation (2), we have

$$(4) \qquad C_3^0 = 1 - \frac{(\omega - 2\Delta)}{\pi} \int \left(\frac{2}{3} f_\Lambda^2 \right) \frac{k'^3 \, d\omega'}{(\omega' - 2\Delta)^2 (\omega' - \omega)} -$$

$$- \frac{(\omega - \Delta)}{\pi} \int \left(-\frac{4}{3} f_\Sigma^2 \right) \frac{k'^3 \, d\omega'}{(\omega' - \Delta)^2 (\omega' - \omega)} = 1 - (\omega - 2\Delta) 2 f_\Lambda^2 I + (\omega - \Delta) 4 f_\Sigma^2 I \,,$$

where $I = (1/3\pi) \int k'^3 \, d\omega' / \omega'^2 (\omega' - \omega)$ and we have put $\Delta = 0$ inside the integral, following ASV. Comparing our results with that of ASV, we shall find that C_3^0 is exactly D_3^0 of ASV.

Now, using eq. (8), we get

$$(5) \qquad \operatorname{tg} \delta_3^0 = \frac{2}{3} \frac{k^3}{D_3^0} \left[\frac{f_\Lambda^2}{\omega - 2\Delta} - \frac{2 f_\Sigma^2}{\omega - \Delta} \right] \qquad (\operatorname{Re} T^{-1} = k \cot \delta) \,,$$

which is the same as that of ASV, if we neglect quantities of the order $4f^4 I\Delta$ in the numerator.

Next, we consider $I = 1$, $J = \frac{3}{2}$ p-wave. Now, we have two channels, and the Born approximations are given by

$$(6) \qquad \begin{cases} B_{\Sigma\Sigma}^3 = \dfrac{2}{3} k_\Sigma^2 \left[-\dfrac{f_\Lambda^2}{\omega - 2\Delta} + \dfrac{f_\Sigma^2}{\omega - \Delta} \right], \\ B_{\Sigma\Lambda}^3 = -\dfrac{2\sqrt{2}}{3} k_\Sigma k_\Lambda \dfrac{f_\Sigma f_\Lambda}{\omega}, \\ B_{\Lambda\Lambda}^3 = \dfrac{2}{3} k_\Lambda^2 \dfrac{f_\Lambda^2}{\omega + \Delta}. \end{cases}$$

Correspondingly, we have the following dispersion representations:

$$(7) \qquad \begin{cases} C_{\Sigma\Sigma}^3 = 1 + (\omega - 2\Delta) 2 f_\Lambda^2 I - 2(\omega - \Delta) f_\Sigma^2 I \,, \\ C_{\Sigma\Lambda}^3 = C_{\Lambda\Sigma}^3 = 2\sqrt{2} f_\Sigma f_\Lambda \omega I \,, \\ C_{\Lambda\Lambda}^3 = 1 - 2(\omega + \Delta) f_\Lambda^2 I \,. \end{cases}$$

From (3), we find that the elements of the K-matrix contain Det C in the denominator. From (7), we come out with the result that Det C^3 is exactly

equal to D_3^1 of ASV. We now have

$$(8)\qquad K_{\Sigma\Sigma}^3 = \frac{2}{3}\frac{k_\Sigma^3}{D_3^1}\left[\left(-\frac{f_\Lambda^2}{\omega-2\Delta}+\frac{f_\Sigma^2}{\omega-\Delta}\right)(1-(\omega+\Delta)\cdot 2f_\Lambda^2 I)+4f_\Sigma^2 f_\Lambda^2 I\frac{k_\Lambda}{k_\Sigma}\right].$$

If we, again, neglect quantities of the order $4f^4 I\Delta$ in our numerator as well as in that of ASV, we arrive at the same result. This happens for all the other K-matrix elements, so that we shall just write down the Born matrix and the corresponding dispersion representation taken by us.

$$\begin{aligned}
I &= 2\,, \qquad J=\tfrac{3}{2}\,, \qquad l=1\,,\\
B_3^2 &= \frac{2}{3}k_\Sigma^2\left(\frac{f_\Lambda^2}{\omega-2\Delta}+\frac{f_\Sigma^2}{\omega-\Delta}\right),\\
C_3^2 &= 1-2(\omega-2\Delta)f_\Lambda^2 I-2(\omega-\Delta)f_\Sigma^2 I\,. \quad (^*)
\end{aligned}$$

$$\begin{aligned}
I &= 0\,, \qquad J=\tfrac{1}{2}\,, \qquad l=1\,,\\
B_1^0 &= -\frac{k_\Sigma^2}{3}\left(\frac{f_\Lambda^2}{\omega-2\Delta}+\frac{9f_\Lambda^2}{\omega}-\frac{2f_\Sigma^2}{\omega-\Delta}\right),\\
C_3^0 &= 1+(\omega-2\Delta)f_\Lambda^2 I+9\omega f_\Lambda^2 I-2(\omega-\Delta)f_\Sigma^2 I\,.
\end{aligned}$$

$$\begin{aligned}
I &= 1\,, \qquad J=\tfrac{1}{2}\,, \qquad l=1\,,\\
B_{\Sigma\Sigma}^1 &= \frac{k_\Sigma^2}{3}\left(\frac{f_\Lambda^2}{\omega-2\Delta}-\frac{7f_\Sigma^2}{\omega-\Delta}\right),\\
C_{\Sigma\Sigma}^1 &= 1-(\omega-2\Delta)f_\Lambda^2 I+7(\omega-\Delta)f_\Sigma^2 I\,,\\
B_{\Lambda\Sigma}^1 &= \frac{\sqrt{2}}{3}k_\Sigma k_\Lambda\left(\frac{1}{\omega}-\frac{3}{\omega-\Delta}\right)f_\Sigma f_\Lambda\,,\\
C_{\Lambda\Sigma}^1 &= \sqrt{2}\,f_\Sigma f_\Lambda I\{-\omega+3(\omega-\Delta)\}\,,\\
B_{\Lambda\Lambda}^1 &= -\frac{k_\Lambda^2}{3}f_\Lambda^2\left(\frac{1}{\omega+\Delta}+\frac{3}{\omega-\Delta}\right),\\
C_{\Lambda\Lambda}^1 &= 1+(\omega+\Delta)f_\Lambda^2 I+3(\omega-\Delta)f_\Lambda^2 I\,,
\end{aligned}$$

If we calculate Det C^1, we again get the result that this is exactly equal to D_1^1 of ASV.

$$\begin{aligned}
I &= 2\,, \qquad J=\tfrac{1}{2}\,, \qquad l=1\,,\\
B_1^2 &= -\frac{k_\Sigma^2}{3}\left(\frac{f_\Lambda^2}{\omega-2\Delta}+\frac{f_\Sigma^2}{\omega-\Delta}\right),\\
C_1^2 &= 1+(\omega-2\Delta)f_\Lambda^2 I+(\omega-\Delta)f_\Sigma^2 I\,.
\end{aligned}$$

(*) C_3^2, C_3^0 and C_1^2 are respectively equal to D_3^2, D_3^0 and D_1^2 of ASV.

Summarizing, we may say that if we neglect quantities of the order of $4f^4 I\Delta = f^2 \Delta/\Omega$ compared to f^2 in our numerators, we get the same equations as those of ASV. Since our denominators are exactly equal to those of ASV, so the position of any resonance predicted by them will also be given by our procedure at the same energy.

3. – Effect of the $\overline{\mathrm{K}}\mathcal{N}$ channel on the position of $I=1$, $J=\frac{3}{2}$ Y-π resonance.

Recent experimental analysis (2) of pion spectra in the reaction $\mathrm{K}^- + \mathrm{p} \to \to \Lambda + \pi^+ + \pi^-$ seems to indicate a $\Lambda\pi$ resonant state in $I = 1$. It has already been pointed out by CAPPS (1) as well as by AMATI, STANGHELLINI and VITALE (2) that for $\frac{1}{2} f_\Sigma^2 \leqslant f_\Lambda^2 \leqslant 2 f_\Sigma^2$, a π-Y p-wave resonance is possible only for $J = \frac{3}{2}$ and $I = 1$ or 2. The position of the $I = 1$ resonance, given by AMATI *et al.* (2) is

$$\omega_r^{(1)} = \Omega - \frac{\Delta}{2} - \frac{5}{6}\delta\Delta \qquad \delta = \frac{f_\Lambda^2 - f_\Sigma^2}{f_\Lambda^2 + f_\Sigma^2}. \tag{9}$$

It is, therefore, interesting to see how far the position of this resonance is affected by the presence of the $\overline{\mathrm{K}}\mathcal{N}'$ channel. This resonance can show up in the $\mathrm{K}^- + \mathrm{p} \to \mathrm{Y} + \pi$ production process (4,7).

The other resonance ($I = 2$) occurring at

$$\omega_r^{(2)} = \Omega + \tfrac{3}{2}\Delta + \tfrac{1}{2}\delta\Delta$$

shows up only in processes like

$$\mathrm{K}^- + \mathrm{p} \to \Sigma^0 + \pi^+ + \pi^- .$$

To investigate the effect of the $\overline{\mathrm{K}}\mathcal{N}'$ channel on the position of the $I = 1$ resonance, we have to calculate Det C taking into account the $\overline{\mathrm{K}}\mathcal{N}'$ channel. In this case, the K-matrix is a 3 by 3 matrix

The Born approximations are

$$\begin{cases} B_{\mathcal{N}\mathcal{N}} = 0 , \\ B_{\mathcal{N}\Sigma} = \dfrac{4}{3}\left(\dfrac{m_\mathrm{Y}}{m_\mathcal{N}}\right)^{\frac{1}{2}} k_\mathcal{N} k_\Sigma \dfrac{f_\mathcal{N} g_\Sigma}{\omega - \Delta} , \\ B_{\mathcal{N}\Lambda} = \dfrac{2\sqrt{2}}{3}\left(\dfrac{m_\mathrm{Y}}{m_\mathcal{N}}\right)^{\frac{1}{2}} k_\mathcal{N} k_\Lambda \dfrac{f_\mathcal{N} g_\Lambda}{\omega} . \end{cases} \tag{10}$$

(7) M. ROSS: *Phys. Rev.*, **112**, 986 (1958); Y. NOGAMI: *Progr. Theor. Phys.*, **22**, 25 (1959).

Correspondingly,

$$C_{\mathcal{N}\mathcal{N}} = 1\,,$$

$$C_{\mathcal{N}\Sigma} = -(\omega - \Delta)4g_{\Sigma}f_{\mathcal{N}}I_1\,,$$

$$C_{\mathcal{N}\Lambda} = -\omega 2\sqrt{2}\,g_{\Lambda}f_{\mathcal{N}}I_1\,. \tag{11}$$

where

$$I_1 = \frac{1}{3\pi}\int_{\omega_T}\frac{k'_{\mathcal{N}}k'^2\mathrm{d}\omega'}{\omega'^2(\omega'-\omega)}\,,$$

ω_T = threshold of the $\bar{\mathrm{K}}\mathcal{N}$ channel; g_{Σ}, g_{Λ} are the $\mathrm{K}\Sigma$, $\mathrm{K}\Lambda$ coupling constants,

$$C_{\Sigma\mathcal{N}} = -(\omega - \Delta)4g_{\Sigma}f_{\mathcal{N}}I_2\,,$$

$$C_{\Lambda\mathcal{N}} = -\omega 2\sqrt{2}\,g_{\Lambda}f_{\mathcal{N}}I_2\,,$$

where

$$I_2 = \frac{1}{3\pi}\int_{\omega_T}\frac{k'^2_{\mathcal{N}}k'\,\mathrm{d}\omega'}{\omega'^2(\omega'-\omega)}\,,$$

We now have, for $J=\frac{3}{2}$, $I=1$,

$$\mathrm{Det}\,C = D_3^1 - 16(\omega-\Delta)^2 g_{\Sigma}^2 f_{\mathcal{N}}^2 I_1 I_2 - 8\omega^2 f_{\mathcal{N}}^2 g_{\Lambda}^2 I_1 I_2\,, \tag{12}$$

neglecting other higher order terms. Here D_3^1 is that given by ASV. One now comes out with the result that the position of the $(J=\frac{3}{2},\ I=1)$ resonance is given by

$$\omega_r^{(1)} = \frac{3[\Omega - \Delta/2 - \frac{5}{6}\delta\Delta] + 4\Delta c_{\Sigma}}{3 + 2c_{\Sigma} + c_{\Lambda}}\,, \tag{13a}$$

where

$$c_{\Sigma} = \left(\frac{g_{\Sigma}^2 f_{\mathcal{N}}^2}{f^4}\cdot\frac{I_1 I_2}{I^2}\right), \qquad c_{\Lambda} = \left(\frac{g_{\Lambda}^2 f_{\mathcal{N}}^2}{f^4}\cdot\frac{I_1 I_2}{I^2}\right), \qquad f^2 = \frac{1}{2}(f_{\Lambda}^2 + f_{\Sigma}^2)\,.$$

We can rewrite (13*a*) in the following form

$$\omega_r^{(1)} = \Omega - \frac{\Delta}{2} - \frac{5}{6}\delta\Delta - \frac{2c_{\Sigma}}{3}\left(\Omega - \frac{5}{2}\Delta\right) - \frac{c_{\Lambda}}{3}\left(\Omega - \frac{\Delta}{2}\right). \tag{13b}$$

The last two terms represent the effect of the $\bar{\mathrm{K}}\mathcal{N}$ channel. To estimate c_{Σ} and c_{Λ}, we have evaluated the integrals I_1 and I_2 using a cut-off at baryon mass and

putting $\omega = 0$ inside the integral ([3]). Also, we have taken the effective ranges for pion-hyperon and pion-nucleon resonance to be equal, as a first approximation. This gives

$$f^2 I \simeq f^2_{\mathcal{N}} \frac{1}{\pi} \int_1^{m_{\mathcal{N}}} \frac{k'^3 \mathrm{d}\omega'}{\omega'^3} .$$

We then find

$$c_\Sigma \simeq \frac{1}{20} \left(\frac{g^2_\Sigma}{f^2_{\mathcal{N}}} \right) . \qquad c_\Lambda \simeq \frac{1}{20} \left(\frac{g^2_\Lambda}{f^2_{\mathcal{N}}} \right) .$$

This shows that even if the $\bar{\mathrm{K}}\mathcal{N}$ coupling constant is comparable to that of the $\pi\mathcal{N}$, the correction due to the $\bar{\mathrm{K}}\mathcal{N}$ channel is small ($\leqslant 5$ MeV).

4. – Discussion.

The Chew-Mandelstam technique ([8]) of finding the scattering amplitude is to write $T = N/D$, where N contains the left-hand cut and D contains the right-hand cut. The FMS technique is to write a dispersion relation for BT^{-1} using the analyticity property of partial wave amplitudes. In the case where N is approximated by the Born term ([9]) and we keep only the right-hand physical cut, both these techniques give the same result.

A complete dispersion relation for $B_{l\pm}$ $(T_{l\pm})^{-1}$ or for that matter, for any $T_{l\pm}$, in the $S(= W^2)$ plane should involve not only the first Riemann sheet $(\sqrt{S} = + W)$ but also the second Riemann sheet $(\sqrt{S} = - W)$. This is evident, when we write a dispersion relation in the W-plane, where we have not only the physical right-hand cut, but also a « left-hand physical cut » ([10]). In the S-plane this « left-hand physical cut » goes to a right-hand cut on the second Riemann sheet. Of course, this cut is related to the right-hand cut on the first sheet by the reflection principle ([11]),

$$f_{l+}(S_{\mathrm{I}}) = - f_{(l+1)-}(S_{\mathrm{II}}) \qquad \sqrt{S_{\mathrm{I}}} = + W, \ \sqrt{S_{\mathrm{II}}} = - W.$$

In all cases we have discussed, the pole of the Born term is near the right-hand cut on the first sheet, so that Born is dominating and to neglect all other

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([8]) G. F. Chew and S. Mandelstam: *Phys. Rev.*, **119**, 467 (1960).
([9]) J. D. Bjorken: *Phys. Rev. Lett.*, **4**, 473 (1960).
([10]) W. R. Frazer and J. R. Fulco: *Phys. Rev.*, **119**, 1420 (1960).
([11]) S. W. MacDowell: *Phys. Rev.*, **116**, 774 (1960).

cuts (including the right-hand cut in the second sheet) can be taken as a first approximation. However, for a partial wave for which the pole of the Born term does not lie near the right-hand cut on the first sheet, as in the S-wave $\bar{K}\mathcal{N}$ processes, this approximation does not work.

* * *

The author is greatly indebted to Dr. P. T. Matthews for many valuable discussions and for guidance. He wishes to express his gratitude to Professor A. Salam for his constant interest and helpful advice. A Commonwealth Scholarship by the Royal Commission for the Exhibition of 1851 is gratefully acknowledged.

Appendix I

Capps has investigated the possibility of a $J=\frac{3}{2}$, $I=1$ resonance in pion hyperon scattering by considering orthogonal combinations of $\pi - \Sigma$ and $\pi - \Lambda$ states. These combinations diagonalize the Born amplitude. In our case, if the Born amplitude is diagonalized then C' ($=BT^{-1}$) is diagonalized, so that $T=B/C'$ is also diagonalized.

These combinations are [12]

$$\psi_r = \frac{1}{(f_\Sigma^2+2f_\Lambda^2)^{\frac{1}{2}}}\left[\sqrt{2}f_\Lambda\psi_{1\Lambda,1+} - f_\Sigma\psi_{1\Sigma,1+}\right],$$

$$\psi_s = \frac{1}{(f_\Sigma^2+2f_\Lambda^2)^{\frac{1}{2}}}\left[f_\Sigma\psi_{1\Lambda,1+} + \sqrt{2}f_\Lambda\psi_{1\Sigma,1+}\right].$$

The Born approximation for the scattering in these two states is given by

$$B_r = \frac{1}{2}(f_\Lambda^2+f_\Sigma^2)\frac{4}{3}\frac{k^2}{\omega},$$

$$B_s = -f_\Lambda^2\frac{2}{3}\frac{k^2}{\omega}.$$

Using eq. (1) we now get [3,4],

$$\frac{\lambda_\alpha k^3\cot\delta_\alpha}{\omega} = 1-\omega r_\alpha,$$

[12] Our combinations have opposite signs in the middle to those of Capps, because our non-diagonal Born element $B_{\Sigma\Lambda}$ has opposite sign.

where

$$r_\alpha = \frac{\lambda_\alpha}{\pi} \int \frac{k'^3 \mathrm{d}\omega'}{\omega'^2(\omega' - \omega)} ,$$

$$\lambda_\alpha = \tfrac{1}{2}(f_\Lambda^2 + f_\Sigma^2)\cdot\tfrac{4}{3} , \qquad \text{for } \alpha = r,$$

$$= -\tfrac{2}{3} f_\Lambda^2 , \qquad \text{for } \alpha = s.$$

We find that for $\alpha = r$, r_α is positive, so that we can expect a resonance at $\omega = 1/r_\alpha$. For $\alpha = s$, r_α is $-ve$; so there does not occur any resonance.

Putting $f_\Lambda = f_\Sigma = f$, we find that the scattering in the state ψ_r corresponds to the (3, 3) pion-nucleon scattering and the resonance is just the analog of he (3, 3) pion-nucleon resonance ([13]). However, it is worth noting that its appearance does not depend on the assumption of global symmetry. The scattering in the state ψ_s corresponds to (1, 3) pion-nucleon amplitude. We shall now obtain the (3, 1) and (1, 1) amplitudes for π-Y scattering.

Assuming $f_\Lambda = f_\Sigma = f$, the (3, 1) and (1, 1) π-Y states are given by ([13])

$$\psi_{r'} = \frac{1}{\sqrt{3}} [\sqrt{2}\psi_{1\Lambda,1-} - \psi_{1\Sigma,1-}] ,$$

$$\psi_{s'} = \frac{1}{\sqrt{3}} [\psi_{1\Lambda,1-} + \sqrt{2}\psi_{1\Sigma,1-}] .$$

We again get

$$\frac{\lambda_\alpha k^3 \cot \delta_\alpha}{\omega} = 1 - \omega r_\alpha ,$$

where for

$$\alpha = r', \qquad \lambda_{r'} = -\tfrac{2}{3} f^2 ,$$

$$\alpha = s', \qquad \lambda_{s'} = -\tfrac{8}{3} f^2 ,$$

and r_α is given by the same expression as before. $\alpha = r'$ corresponds to (3, 1) amplitude and $\alpha = s'$ to (1, 1) amplitude.

Appendix II

For the two-channel case, we may write ([14]),

$$T^{-1} = \begin{pmatrix} a - ik_1 & c \\ c & b - ik_2 \end{pmatrix}, \tag{A2.1}$$

([13]) D. Amati and B. Vitale: *Nuovo Cimento*, **9**, 895 (1958); M. Gell-Mann: *Phys. Rev.*, **106**, 1297 (1957).

([14]) P. T. Matthews and A. Salam: *Nuovo Cimento*, **13**, 381 (1959). A detailed discussion on the K-matrix formalism has been given by R. H. Dalitz and S. F. Tuan: *Ann. Phys.*, **10**, 307 (1960).

when both the channels are open. The K-matrix, in this case, is given by

$$k^{\frac{1}{2}}K^{-1}k^{\frac{1}{2}} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}. \tag{A2.2}$$

If we now define the condition for resonance as the vanishing of the denominator of the K-matrix elements, as has been used by ASV, we have

$$ab - c^2 = 0\,. \tag{A2.3}$$

However, there are two other ways of defining the condition for resonance. One is to take

$$\mathrm{Re}\,(\mathrm{Det}\,T^{-1}) = 0\,,$$

which gives

$$ab - c^2 - k_1 k_2 = 0\,, \tag{A2.4}$$

while the other is

$$\mathrm{Re}\left(\frac{1}{T_{ij}}\right) = 0\,, \tag{A2.5}$$

which gives the position of resonance for the « 11 » process as

$$\begin{cases} \text{(i)} \quad (ab - c^2)b + k_2^2 a = 0, \\ \text{for the « 12 » or « 21 » process as} \\ \text{(ii)} \quad ab - c^2 - k_1 k_2 = 0\,, \\ \text{and for the « 22 » process as} \\ \text{(iii)} \quad (ab - c^2)a + k_1^2 b = 0\,. \end{cases} \tag{A2.6}$$

If we neglect the momentum dependent terms, we find that (A2.3), (A2.4) and (A2.6) are the same.

Let us now suppose that channel 2 is closed. Then, if we completely ignore this channel, the condition for resonance for the « 11 » process is

$$a = 0\,. \tag{A2.7}$$

If, on the other hand, we take into account the presence of channel 2, by making the continuation $k_2 \to i|k_2|$, the condition for resonance becomes, in the K-matrix formalism,

$$a(b + |k_2|) - c^2 = 0\,. \tag{A2.8}$$

If we neglect the momentum dependent term, we get,

$$ab - c^2 = 0\,. \tag{A2.9}$$

If the coupling between channels 1 and 2 is weak, then « c » is small, so that (A2.8) or (A2.9) gives essentially $a \approx 0$ as the condition for resonance. For strongly coupled channels, however, we can expect considerable deviation from $a \approx 0$.

RIASSUNTO (*)

La tecnica terorica di dispersione, suggerita da Feldman, Matthews e Salam, è stata usata per derivare le equazioni statiche per lo scattering pione-iperone, e si dimostra che l'inclusione del canale $\bar{K}\mathcal{N}'$ non cambia le conclusioni sulla risonanza π-Y, con $J = \frac{3}{2}$, $I = 1$, dell'onda p, esposte da Amati, Stanghellini e Vitale e da Capps.

(*) *Traduzione a cura della Redazione.*

Polarization of $\bar{\mathrm{p}}$ in $\bar{\mathrm{p}}C$ Scattering (*).

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(ricevuto il 23 Dicembre 1960)

It has been shown by Sakurai (1) that it is possible to explain the repulsive core and the spin-orbit force in nucleon-nucleon scattering, by the introduction of a vector mesonlike 3π resonance in $T=0$ and $J=1$ state. We use the parameters obtained in this analysis to estimate the polarization of antiprotons scattered from a carbon target. The parameters are, mass of 3π resonance $=3$ pion masses and coupling constant $f^2/4\pi=3.5$. The Wolfenstein (2) C amplitude can now be written down for one-vector-meson exchange graph of $\bar{\mathrm{p}}\mathrm{N}$ scattering.

$$C = -i\,\frac{f^2}{4\pi}\,\frac{3}{2E}\left\{1+\frac{k^2 x}{3(E+m)^2}\right\}\frac{\sin\theta}{x_0-x},$$

and

$$x_0 = 1+\frac{\mu^2}{2k^2},$$

where k and E are the antiproton momenta and energy, in the c.m. system of $\bar{\mathrm{p}}\mathrm{N}$, respectively, and μ is the mass of the conjectured 3π resonance.

We now use the optical model to calculate the scattering amplitude of $\bar{\mathrm{p}}$ from the C nucleus. The central potential is gotten from experimental analysis (3), and the spin-orbit potential is calculated (4) from the known C amplitude. If

$$V^{\pm} = -(u+iw)\varrho(r) + V_{LS}\,\frac{1}{\mu_\pi^2}\,\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\,\varrho(r)\begin{Bmatrix} l \\ -(l+1)\end{Bmatrix},$$

(*) Work carried out under the auspices of the U.S. Atomic Energy Commission.
(1) J. Sakurai: *Phys. Rev.*, **119**, 1784 (1960).
(2) L. Wolfenstein: *Phys. Rev.*, **96**, 1654 (1954).
(3) A. G. Glassgold: *Prog. Nucl. Phys.*, **7**, 124 (1959).
(4) W. B. Rosenfeld and K. M. Watson: *Phys. Rev.*, **102**, 1157 (1956).

then

$$V_{LS} = -Q\Gamma M_1(O)\,,$$

where

$$Q = \left[\frac{2E_{\mathrm{c.m.}}}{M}\right]\left[\frac{\mu_\pi}{p}\right]^2,$$

$$\Gamma = \frac{3}{\lambda^3}\mu_\pi^2\left[\frac{E_{\mathrm{c.m.}}}{E_{\mathrm{lab}}}\right]\cdot\frac{\mu_\pi}{M},$$

$$M_1(\theta) = -\frac{1}{2i}\,\frac{C_{\bar{\mathrm{n}}\mathrm{n}}(\theta) + C_{\bar{\mathrm{p}}\mathrm{p}}(\theta)}{\sin\theta}\,.$$

and

$$R_A = \frac{\lambda\!\cdot\!1^{\frac{1}{3}}}{\mu_\pi}\,.$$

where R_A is the radius of the nucleus, μ_π the mass of the pion and M the mass of the nucleus.

If we use the square well for the nuclear shape,

$$V^{\pm} = -(u+iw)\theta(R-r) - \frac{V_{LS}}{\mu_{\pi^2}}\,\frac{\delta(r-R)}{R}\begin{Bmatrix} l \\ -(l+1)\end{Bmatrix}.$$

The phase shifts are then [5],

$$2\delta^{\pm} = \frac{(u+iw)}{T}\,kRS_l + \frac{V_{LS}}{\mu_{\pi^2}}\,\frac{k}{T}\cdot\frac{1}{R}\,S_l^{-1}\begin{Bmatrix} l \\ -(l+1)\end{Bmatrix} \qquad l < kR - \frac{1}{2}\,,$$

$$= 0 \qquad l \geq kR - \tfrac{1}{2}\,.$$

where

$$T = \frac{k^2}{2M}\,, \qquad S = \left\{1 - \left(\frac{l+\frac{1}{2}}{kR}\right)\right\}^{\frac{1}{2}},$$

k is the c.m. momentum of $\bar{\mathrm{p}}$ in the $\bar{\mathrm{p}}C$ c.m. system.

The scattering amplitude is now

$$f(\theta) = g(\theta) + \sigma\cdot n h(\theta)\,,$$

where

$$g(\theta) = \frac{1}{2ik}\sum_0^\infty\left[(l+1)\eta_l^+ + l\eta_l^-\right]P_l(\cos\theta)\,,$$

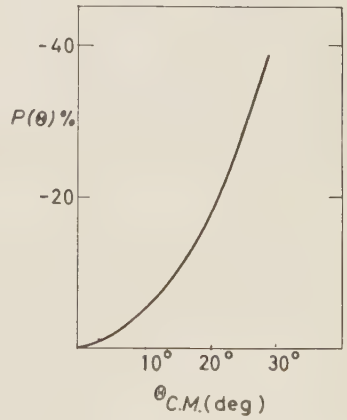

Fig. 1. – $\bar{\mathrm{p}}$ polarization $\bar{\mathrm{p}}C$ scattering at 160 MeV/c.

[5] S. Fernbach, W. Hechrotte and L. V. Lepore: *Phys. Rev.*, **97**, 1059 (1955).

and

$$h(\theta) = -\frac{1}{2k}\sum_{0}^{\infty}[\eta_l^+ - \eta_l^-]\frac{\mathrm{d}}{\mathrm{d}\theta}P_l(\cos\theta)\,,$$

$$n = \frac{\boldsymbol{k}_{\mathrm{in}}\cdot\boldsymbol{k}_{\mathrm{out}}}{\boldsymbol{k}_{\mathrm{in}}\cdot\boldsymbol{k}_{\mathrm{out}}}\,.$$

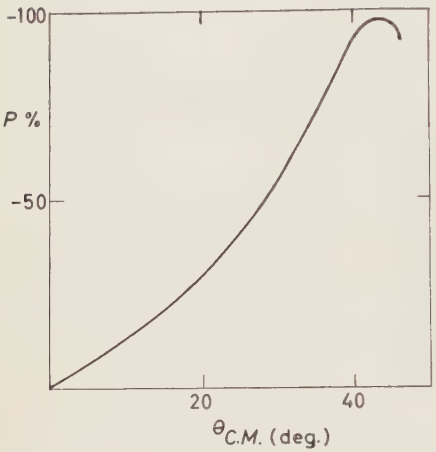

Fig. 2. – $\bar{\mathrm{p}}$ polarization in $\bar{\mathrm{p}}$p scattering at 1.6 GeV/c lab. momenta.

Polarization is given by

$$p(\theta) = \frac{2\ \mathrm{Re}\ gh^*}{|g|^2 + |h|^2}\,.$$

When calculation is carried out for $\bar{\mathrm{p}}C$ for lab. energy of 160 MeV, we get an average polarization of 11.4%.

Estimate of $\bar{\mathrm{p}}$p polarization.

In $\bar{\mathrm{p}}$p system as both particels have spin, the scattering matrix has more terms and we can now write

$$PI_0 = 2\ \mathrm{Re}\ C^*(A + B)\,,$$

where C and A are the spin orbit and central terms respectively and B is the coefficient of $\sigma_{\mathrm{p}}\cdot n\sigma_{\mathrm{p}}\cdot n$. If we assume, just to obtain an estimate, that $B \ll A$ for forward angles, we can again use the above model to estimate the polarization. We get on calculation, at 1.6 GeV/c lab. momenta, an average polarization of 39%, between 5 and 20 degrees lab. angle. Considering the crudeness of this estimate, it may not be inconsistent with the experimental value ([6]) of (0.48 ± 0.95) between $6° < \theta < 26°$. It should be noted that the spin orbit term is purely real.

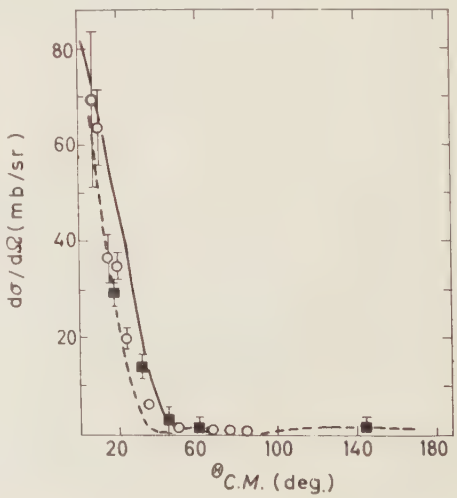

Fig. 3. – Differential cross section of $\bar{\mathrm{p}}$p at 1.6 GeV/c momentum. ▪ Cork, Lambarston, Wentzel, Armenteros and Coebes; ○ B. C. Maglić. Solid line is on our model; dashed line is due to Grieder and Glassgold.

* * *

The author wishes to thank Professor J. Sakurai for suggesting the problem and for help throughout the course of this work.

([6]) B. C. Maglić: UCRL-9336 (unpublished).

Observation of a Hyperfragment Decay by π^0 Emission.

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(ricevuto il 27 Dicembre 1960)

The Λ-hyperon decays by 2 major modes

$$\Lambda \to \pi^- + \mathrm{p}\,, \qquad \Lambda \to \pi^0 + \mathrm{n}\,,$$

with a branching ratio of about 2:1. Many bound hyperon decays of the first mode have been observed. The second mode could be detected in a number of ways. The real photons from

$$\pi^0 \to \gamma + \gamma\,,$$

may convert near the point of decay. This is unlikely because the conversion length in emulsion is about 3.97 cm, and has never been observed. Ammar [1] has presented statistical evidence for the existence of

$$^4\mathrm{He}_\Lambda \to \pi^0 + {}^4\mathrm{He}\,.$$

With this mode a recoil of unique range 8.9 μm in emulsion is expected, so that the hypernucleus and recoil resemble a prong with a scatter. Ammar observed the number of recoils between 8 to 10 μm on prongs of $Z=2$ from K^--stars. He found there to be more than could be expected as a result of scattering.

Another possibility for detecting the π^0 decay of the bound hyperon is if the π^0 decays with internal conversion of one of the γ's: this mode is only 1/80 times as frequent as the 2γ mode. Levi-Setti and Slater [2] have reported one such event from a $Z=2$ hypernucleus according to

$$^4\mathrm{He}_\Lambda \to \pi^0 + {}^4\mathrm{He}\,,$$

or

$$^{4,5}\mathrm{He}_\Lambda \to \pi^0 + \mathrm{n} + {}^{3,4}\mathrm{He}\,.$$

An event somewhat similar has been found in an emulsion stack of 240, .6 mm, 6 in. × 10 in. Ilford K-5 pellicles exposed to the enriched K^- beam of the Bevatron in January, 1959. A projection drawing of the event appears in Fig. 1. At *A* a K^- is captured at rest, a Σ^- is emitted which comes to rest after 1.24 mm and is captured at *B*. This gives a steep prong of 99.5 μm and a short 5 μm prong. The steep prong is identified as a hypernucleus which terminates at *C* in a flat stub (*R*) 2.4 μm long and a pair of minimum tracks (e_1 and e_2) diverging at 14°.

(1) R. G. Ammar: *Nuovo Cimento*, **14**, 1226 (1959).

(2) R. Levi-Setti and W. E. Slater: *Phys. Rev.*, **111**, 1395 (1958).

e_1 and e_2 were identified as electrons by ionization and scattering measurements

$$g^*(e_1) = 0.96 \pm 0.09 ,$$
$$g^*(e_2) = 0.94 \pm 0.05 ,$$
$$p\beta c(e_1) = (20.7 \pm 4) \text{ MeV} ,$$
$$p\beta c(e_2) = (93.6 \pm 10) \text{ MeV} .$$

scattering displacement is 0.25 μm for the slow electron. These lines lie within 0.1 μm from the centre of each grain and give an intersection at C. The Σ^- is identified as such by integral gap and scattering measurements. These give a mass of (1.8 ± 0.6) proton masses compared with the accepted value of 1.3.

Table I contains the ranges, energies

Fig. 1. – Projection drawing of a hyperfragment decay by π^0 emission.

The scattering was performed over 1 mm and 5 mm, the grain count over 0.5 mm and 2 mm respectively, all in the same plate as the event. A correction to $p\beta c$ for ionization loss over the track used has been made. To detect whether the electrons diverge from C, drawings have been made using a projection microscope and the vertical separation of one track from the other measured. The latter gives the more sensitive test because the divergence is mostly in the vertical plane with one electron very flat. Best line fits were made to the first 25 μm of track, over which distance the mean and directions of the prongs from the hypernuclear decay and Σ^--capture star.

It is impossible to fit the hypernuclear decay kinematics by a 2-body π^0-decay because the electron pair have high energy and are close in direction to the recoil (R). If we allow a single neutron then the decays

$$^{3,4}\mathrm{H}_\Lambda \rightarrow \mathrm{n} + \pi^0 + {}^{2,3}\mathrm{H} ,$$
$$^{4,5}\mathrm{He}_\Lambda \rightarrow \mathrm{n} + \pi^0 + {}^{3,4}\mathrm{He} ,$$

are possible interpretations. The momentum of the recoil is poorly determined except for $^{4,5}\mathrm{He}_\Lambda$. For these the range-momentum curve at short

TABLE I. – *Description of stars.*

Particle	Range or energy	Dip	Azimuth
e_1	(20.7 ± 4) MeV	10° 43′ ± 31′	3° 6′ ± 6′
e_2	(93.6 ± 10) MeV	− 2° 40′ ± 35′	0°
Recoil (R)	(2.4 ± 0.2) mμ	0° ± 2° 18′	21° 6′ ± 2°
Hypernucleus	(99.5 ± 1) mμ	81° 30′ ± 10′	0°
Recoil	(4.9 ± 0) mμ	− 38° 56′ ± 7′	140° ± 2°

ranges can be normalized to fit decays of the type

$$\text{Hypernucleus} \to \pi^- + \text{p} + \text{recoil},$$

which are quite common for $^{4,5}\text{He}_\Lambda$. $^{7,8}\text{Li}_\Lambda$ decaying in the one-neutron mode are also possible assignments but not heavier hyperfragments. On extending the analysis to 2 or 3 neutron modes we can fit in addition $^{7}\text{He}_\Lambda$ and $^{9}\text{Li}_\Lambda$.

As the visible momentum is closely directed to e_2 the allowed directions of the π^0 in the single neutron analysis lie almost on a cone with e_2 as axis. For $^{5}\text{He}_\Lambda$ the angle between the π^0 and e_2 varies between 19° and 22° and the π^0 energy between 23.4 and 24.9 MeV. The opening angle of the electron pair in the rest system is then 23° ± 6′. DALITZ (3) has shown that half the opening angles of internal conversion electrons are < 16° and that the probability in this region falls off as the reciprocal of the opening angle, so 23° is quite reasonable. Similar results follow for the other single neutron interpretations. The single neutron analyses are possible throughout the estimated range of errors for $^{3,4}\text{H}_\Lambda$ and $^{4}\text{He}_\Lambda$. Any considerable radiation loss by either electron would spoil the fit. For the 93.6 MeV/c electron the average radiation loss in 5 mm is calculated to be 18 MeV. No large scatters occur so we expect it to be much less than this.

Some information on the nature of the hypernucleus can be got from the Σ-capture. $Z \geqslant 4$ interpretations are excluded if the capture occurs on a heavy emulsion nucleus and $Z \geqslant 3$ excluded if it occurs on a light emulsion nucleus.

* * *

I wish to thank Prof. D. H. WILKINSON for his encouragement and helpful discussions; also Dr. S. LOKANATHAN and Dr. J. H. MULVEY for reading and commenting on the manuscript.

Best thanks are due to Dr. E. J. LOFGREN and the staff of the Bevatron for making possible the exposure and to the William Waldorf Astor Foundation for supporting it.

(3) R. H. DALITZ: *Proc. Roy. Soc.*, A 64, 667 (1951).

$\beta^+ \to \gamma$ Angular Correlation in the Decay of ^{22}Na.

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(ricevuto il 24 Gennaio 1961)

Recently much interest has been aroused in the study of allowed β-transitions in order to look for weak magnetism contributions [1] and other second forbidden matrix element contributions [2] to the allowed β-decay. Regarding the $\beta^+ \to \gamma$ directional correlation with ^{22}Na, Steffen [3] found an anisotropy (with the selection of positrons of energy 300 keV) of $-(0.0027 \pm 0.0004)$, and Daniel and Eakins [4] obtained an integral ($E_\beta > 100$ keV) correlation anisotropy of $-(0.02 \pm 0.002)$ and differential correlation anisotropies whose energy dependence could not be established definitely. It appeared possible to account for certain contributions due to electron (or positron) scattering and annihilation quanta and thus to obtain more definite results in both differential and integral angular correlation. So, in view of the importance of such results, these measurements have been made.

A fast-slow coincidence system, employing RCA 6810A photomultipliers, with a resolving time of 26 ns was used. The γ-detector was a 1 ½ in. × 1 ½ in. NaI(Tl) crystal with a resolution at 661 keV of about 7.5%. A vacuum chamber of aluminium with a ¼ in. perspex sheet covering the inner surface was used in the β-channel. The chamber was designed with particular attention to minimize scattering on to the detector. The β-detector was a 1 in. diam. × 1/10 in. thick anthracene crystal covered with a 150 μg/cm² aluminium reflector. Resolution with this arrangement was about 15% at 624 keV. In the earlier works [3,4] the scattering from detector to detector which makes large contributions in the 90° and 270° positions have been noted. In order to avoid this, both the detectors were surrouded by suitably graded shields. The γ-detector was provided with a frontal shield of 4 mm lead and 1 mm copper. Source to crystal distance was 11.5 cm in both channels. All the following observations have been corrected for finite solid angle effect by calculating the correction factors by numerical integration of the relevant integrals [5]. Source centering was done to about 0.5% in the counting rate at the different angular positions. Only the photopeak of the 1.28 MeV γ-ray was accepted in the γ-channel.

(1) M. Gell-Mann: *Phys. Rev.*, **111**, 362 (1958).
(2) M. Morita: *Phys. Rev.*, **113**, 2584 (1959); *Nucl. Phys.*, **14**, 106 (1959).
(3) R. M. Steffen: *Phys. Rev. Lett.*, **3**, 277 (1959).
(4) H. Daniel and G. W. Eakins: *Phys. Rev.*, **117**, 1565 (1960).
(5) M. E. Rose: *Phys. Rev.*, **91**, 610 (1953).

High specific activity $^{22}Na_3 Po_4$ was spread over an area of about 0.5 cm² on a 150 μg/cm² aluminium backing which was supported by an aluminium ring. On account of this conducting environment and the short half-life of the intermediate (1.28 MeV) state in ^{22}Ne [being $1.5 \cdot 10^{-12}$ s as calculated from the observed (6) half-life of $5 \cdot 10^{-13}$ s for the 1.632 MeV level in ^{20}Ne], the perturbations of nuclear orientation in the intermediate state should not affect the angular correlation.

In the case of integral correlation measurement, observations were taken at angular positions which were selected at random from intervals of 30° in the range of 90° to 270°. In differential correlation measurements observations were confined to 90°, 180° and 270° positions. True to chance coincidence ratio was always better than 20. Chance coincidences were measured with a 125 ns delay in one of the channels. True coincidence rates were divided by singles counting rates which were regularly checked in order to ensure the stability of the settings. Coincidence rates at the symmetric positions in the two quadrants were also checked. As a further check, the integral $\beta^- \rightarrow \gamma$ angular correlation was observed with ^{60}Co for which (after the necessary corrections) an anisotropy of $-(0.002 \pm 0.012)$, in good agreement with earlier works, was obtained (Fig. 1*a*).

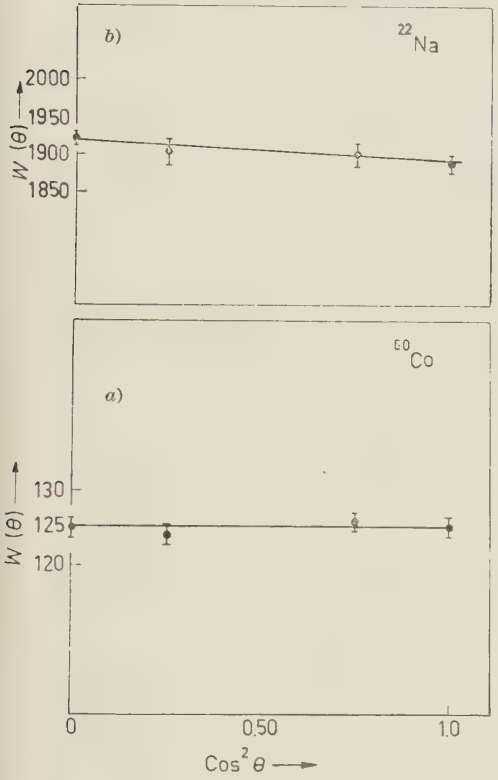

Fig. 1. – *a*) Integral ($E_\beta > 120$ keV) $\beta^- \rightarrow \gamma$ directional correlation with ^{60}Co. *b*) Integral ($E_\beta > 120$ keV) $\beta^+ \rightarrow \gamma$ directional correlation with ^{22}Na.

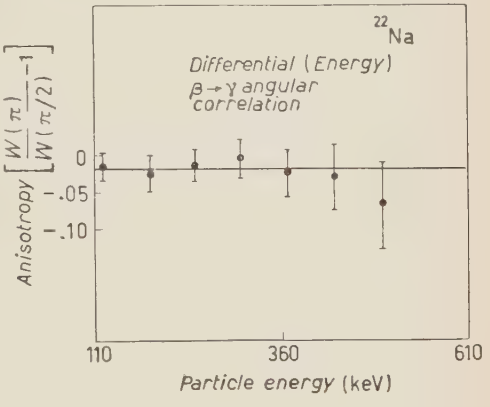

Fig. 2. – Differential (in energy) $\beta^+ \rightarrow \gamma$ directional correlation results. Anisotropy at various energy settings.

The integral $\beta(E_\beta > 120$ keV$) - \gamma(1.28$ MeV) correlation could be fitted by least squares method to

$$W(\theta) = 1 - (0.015 \pm 0.003) P_2(\cos\theta) .$$

(6) S. Devons, G. Manning and J. H. Towle: *Proc. Phys. Soc.*, A 69, 173 (1956).

The coefficient of P_2 has been corrected for finite solid angle effects. In auxilliary runs with the same source as in the main runs but with the source holder placed in the horizontal position and the direct beam of positrons being cut off by means of a stack of narrow insulation tape, coincidences to be called « scattering effects » were determined. These scattering effects include coincidences between scattered positrons and 1.28 MeV γ-ray or photons due to two quanta annihilation or with photons due to annihilation-in-flight which contribute to the photopeak region of the 1.28 MeV γ-ray. They also include annihilation quantum 1.28 MeV γ-coincidences, annihilation quantum (appearing in the Compton continuum in the β-channel) coincidences with the other annihilation quantum which contributes to the photopeak region of the 1.28 MeV γ due to simultaneous detection of pulses in the region of Compton continua of the 1.28 MeV γ-ray and the annihilation quantum. These « scattering effect » contributions were of the order of chance coincidences (about 3% of the true coincidences) and were subtracted out from the true coincidences of the main runs.

The differential $\beta^+ \rightarrow \gamma$ directional correlation was observed in the region of 120 keV to 450 keV in steps of 60 keV. After correcting for finite solid angle effect and finite detector resolution (in energy), the anisotropies observed at different energies are shown in Fig. 2. These values of anisotropies were deduced after correcting the coincidence data for chance coincidences and « scattering effects » which were determined, as described above, for each energy and each angular setting. A least square analysis of these anisotropies at different energies, under the assumption of energy independence, gave $A = 0.018\,4 \pm 0.019\,5$. Least square fit of these anisotropy values to a straight line normalized at the value at 120 keV gave

$$A = -0.015 - (0.018 \pm 0.019)E\,, \qquad E \text{ in MeV}\,.$$

The integral correlation measurements are in agreement with those of Daniel and Eakins [4] but differential correlation measurements indicate that the present values of anisotropies are lower at almost all energies than those reported by them. Further energy dependence in the present case is much lower than that found by them. The anisotropy at 300 keV is in good agreement with that of Steffen [3].

In the case under consideration, anticipating the occurrence of s- and p- or d-wave interference terms, the angular correlation function for the decay scheme

$$^{22}\mathrm{Na}(3+) \xrightarrow{\beta} {}^{22}\mathrm{Ne}^*(2+) \xrightarrow[E_\gamma]{\gamma} {}^{22}\mathrm{Na}(0+)\,,$$

may be written as,

$$W(\theta) = 1 + A_2 P_2(\cos\theta)\,,$$

with

$$A_2 = F_2(1132)F_2(2202)[B_2] = -0.071\,8B_2\,.$$

In this F_2's are the geometrical F coefficients [7] characterizing the angular momenta involved, B_2 is a function of particle parameters characterizing the

[7] K. Alder, B. Stech and A. Winther: *Phys. Rev.*, **107**, 728 (1957).

β-transition. The term due to weak magnetism occurring through the momentum type matrix elements $\mathcal{M}(\boldsymbol{\alpha} \times \boldsymbol{\gamma})$ contribute a value of

$$B_2' = - \frac{\frac{4}{3} \frac{\mu_p - \mu_n}{2Mc^2} \frac{v^2}{c^2}}{1 + \frac{4}{3} \frac{\mu_p - \mu_n}{2Mc^2} \left\{ E_0 - E \left(1 + \frac{v^2}{c^2} \right) \right\}} E ,$$

in the notation of ALDER *et al.* [7] with μ_p and μ_n representing the anomalous magnetic moments of proton and neutron respectively. This leads to $A_2 = +\, 0.064 \cdot 10^{-3}$. As noted above all experimental values reported so far are much larger than this and the sign is opposite. The de Broglie wavelength effect, Coulomb effects and co-ordinate type matrix elements [2] taken into account do not still account for the observed anisotropy.

The β^+-transition has a very large log *ft* value of 7.39. Electron capture to positron emission branching ratio has been determined experimentally [8] to be 0.110 ± 0.006 and shown to be theoretically explicable if the transition is either *GT* allowed or non-unique first forbidden. The large *ft* value indicating incomplete superposition of initial and final state wave functions cannot be attributed to *l*-forbiddingness in as far as the EC/β^+ ratio required is about half the observed value. Spin and magnetic moment [9] determinations favour an allowed $(3^+ \to 2^+)$ β-transition. The shape of the β-spectrum [10] has been shown to have an allowed shape down to about 120 keV and a slight preponderance of low energy positrons below 120 keV as in the case of many other allowed and non-unique first forbidden transitions. Isotopic spin selection rules do not forbid the transition. Thus, the large log *ft* value cannot be explained. From these considerations, it is also seen that T_{ij} matrix elements cannot contribute significantly.

The results of this paper are in essential agreement with those of earlier workers [3,4] and corrections, precautions and checks of the performance of the set-up were made regularly. Therefore, it is concluded that the large (for an *l*-allowed isotopic spin allowed *GT* allowed β-transition) anisotropy cannot be explained even like the large log *ft* value.

* * *

The author is very thankful to Dr. B. V. THOSAR for his interest.

[8] R. SHERR and R. H. MILLER: *Phys. Rev.*, **93**, 1076 (1954).
[9] J. E. MACK: *Rev. Mod. Phys.*, **22**, 64 (1950).
[10] J. H. HAMILTON, L. M. LANGER and W. G. SMITH: *Phys. Rev.*, **112**, 2010 (1958).

IL NUOVO CIMENTO Vol. XX, N. 1 1° Aprile 1961

Application of the Chew and Low Extrapolation Procedure to $K^- + d \to Y + \mathcal{N} + \pi$ Absorption Reactions.

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(ricevuto il 30 Gennaio 1961)

There still exists considerable uncertainty regarding the Dalitz-Tuan [1] (D-T) scattering length parametrizations of the low-energy K^--p scattering data. Each of the four possible scattering length sets has, at some time or another been put forward as a tentative favourite. In this Letter we should like to point out that it may be possible to separate the D-T solutions by a study of the capture reaction $K^- + d \to Y + \mathcal{N} + \pi$ using the Chew and Low [2] extrapolation technique.

In impulse approximation we may graphically represent the reaction

$$K^- + d \to Y + \mathcal{N}_1 + \pi$$

as in Fig. 1. The deuteron is supposed to be at rest. This diagram also serves to illustrate the existence of a pole in the transition amplitude of the process. This pole occurs at $p^2 = -\alpha^2$, where α^2/M is the binding energy of the deuteron (M is the nucleon mass), and has a residue which is determined essentially by the asymptotic normalization of the deuteron and the amplitude, on the mass shell, for the process $K^- + \mathcal{N}_2 \to Y + \pi$. This amplitude may in general correspond either to a physical energy or to an unphysical energy below the $K^-\mathcal{N}$ threshold. If the K^--meson is captured from rest the amplitude corresponds to a momentum of approximately 16 MeV/c below this threshold.

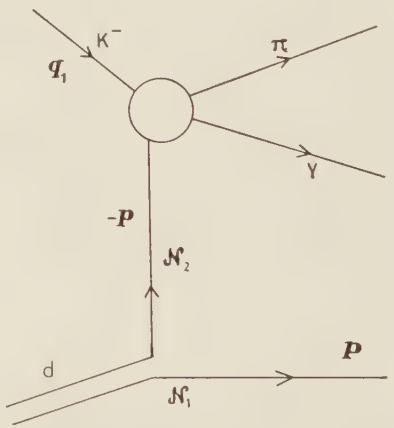

Fig. 1. – Pictorial representation of the process $K^- + d \to Y + \mathcal{N}_1 + \pi$ in impulse approximation.

[1] R. H. Dalitz and S. F. Tuan: *Ann. Phys.*, **3**, 307 (1960).

[2] G. F. Chew and F. E. Low: *Phys. Rev.*, **113**, 1640 (1959).

Further, the $K^- + \mathcal{N} \to Y + \pi$ reaction is sufficiently exothermic for there to be a large range in p even though q_1 is zero. We suggest therefore that the capture at rest data be extrapolated to $p^2 = -\alpha^2$ in order to determine a value for the $K^- + \mathcal{N} \to Y + \pi$ cross-section below the $K\mathcal{N}$ threshold, which can then be compared with the predictions of the various D-T solutions.

The transition rate spectrum, valid at the pole, can be obtained by a modification of the method used in reference (2). We find, on integrating over θ_L, the laboratory angle between $\boldsymbol{p}$ and $\boldsymbol{q}_1$,

$$(1) \qquad \frac{\partial R}{\partial p^2} \xrightarrow{p^2 \to -\alpha^2} 8\pi^2 \frac{p}{(p^2+\alpha^2)^2} |t|^2 .$$

In this paper we use the same notation, *mutatis mutandis*, as in reference (2). (See also Fig. 1). The amplitude t is related to the total $K^- + \mathcal{N}_2 \to \to Y + \pi$ cross-section by

$$(2) \qquad \sigma_{\text{tot}} = \frac{4\pi}{p} |t|^2 .$$

In contrast to the cases discussed by CHEW and LOW (2), ω (the $Y\pi$ c.m. energy) and p^2 are not independent but are directly linked by energy conservation:

$$(3) \qquad \begin{cases} \omega^2 = W^2 + M^2 - \dfrac{W}{M}(p^2 + 2M^2) , \\ W = M_d + m , \end{cases}$$

where m is the mass of the K-meson.

The possibility of a significant extrapolation to the pole at $p^2 = -\alpha^2$ rests on the assumption that any other singularities of the amplitude are insignificant compared with this pole in the extrapolation region. It is thus necessary to investigate the analytical structure of the amplitude of the whole process as a function of ω^2, p^2, and the $K\mathcal{N}Y\pi$ vertex momentum transfer.

It is to be expected that final-state interactions will introduce further singularities into the amplitude. We have investigated, on a simple model, the effect of elastic Y-$\mathcal{N}$ scattering. This leads to branch points near $p = 120$ MeV/c together with a logarithmic branch point at $p^2 = -\alpha^2$. With CHEW and LOW we do not expect these singularities to be important. Since the Y-$\mathcal{N}$ final-state interaction is probably the most important one, multiple-scattering singularities are unlikely to prevent the desired extrapolation.

It is necessary also to investigate the analytic structure of t, which, away form the pole, is off the mass shell. This is because ω^2 is a function of p^2 by (3) and we expect singularities in ω near the extrapolation region.

One possibility is to assume the same form for t off the mass shell as on. In this case, using the Dalitz-Tuan formulae (1), we find a branch point due to the $K^-\mathcal{N}$ threshold at $p^2 = -1.2\alpha^2$. A known singularity at this point can be allowed for in the extrapolation.

From a practical point of view, since no absolute values of the transition rate R have been measured it is necessary either to extrapolate $(p^2+\alpha^2)^2(\partial R/\partial p^2)$ to the pole for two reactions separately and then take their ratio or to extrapolate the ratio of $\partial R/\partial p^2$ for two reactions directly. The latter procedure may be preferable since some of the singularities in $|t|^2$ may cancel and so facilitate the extrapolation. For instance, if we assume, as mentioned previously, that the isotopic spin amplitudes of the $K^- + \mathcal{N} \to Y + \pi$ reaction have the same form off the mass shell as on, then they can be written as $a_T b^T(Y\pi)$, where a_T contains awkward singularities and depends on the isotopic spin T and not on the final channel $Y\pi$. The singularities in a_T will then cancel on taking the ratio of reaction rates in different channels with the same T. Unfortunately the quantities obtained by extrapolating such ratios will not

distinguish between the D-T solutions; for example, $|b^1(\Lambda\pi)|^2/|b^1(\Sigma\pi)|^2$ is the same for all of them and is assumed to be constant in the D-T analysis. To distinguish between the D-T solutions, a ratio involving different isotopic states must be used and the singularities in t are then more important. We defer an investigation of these for a future paper.

A similar analysis could be made of the reaction $K^-+{}^4He \to {}^3I+Y+\pi$ (${}^3I={}^3He$ or ${}^3H$). Due to the increase in binding energy difference the pole occurs much further (62 MeV/c) below the $K^-\mathcal{N}$ threshold than in the K^-+d case. This is approaching the resonance region predicted by some of the D-T solutions and we can therefore expect a more definite separation of these. However, a well-defined pole term may be more difficult to separate out from the total amplitude and the ${}^4He \to {}^3I+n$ vertex Γ probably depends fairly strongly on the momentum p. These two things alone make the extrapolation a rather difficult one, although the variation of Γ with p could be allowed for by calculating it using the AMADO and BLANKENBECLER (3) technique. This variation would show up as structure in the spectator momentum spectrum.

This same vertex appears also in the theory of the $d+{}^3He \to {}^4He+p$ and $d+{}^3H \to {}^4He+n$ stripping reactions. If we take the angular distributions of these reactions to reflect the variation of Γ with p, then the experimental curves indicate that this is not negligible.

An appreciable contribution by the pole term to the physical process could, if there is a π-Λ resonance in the region of the pole, account for the increased Λ production observed in $K^-+{}^4He$ as compared with K^-+d absorption, although it may be possible that a Λ-I^3 resonance is also contributing. Moreover it is probably unwise to compare directly K^-+d and $K^-+{}^4He$ reactions since multiple K-meson scattering and many-body effects are more important in the latter.

* * *

I am deeply indebted to Mr. L. CASTILLEJO for many helpful discussions and much encouragement. I am also grateful to the Department of Scientific and Industrial Research for the award of a Research Studentship.

(3) R. D. AMADO and R. BLANKENBECLER: *New Methods in Direct Interaction Theory*, preprint.

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Theoretical Discussion of Possible Experiments with Electron-Positron Colliding Beams.

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(ricevuto il 2 Febbraio 1961)

1. – We discussed recently the possible determination of the pion form factors from the reactions $e^+ + e^- \to n\pi$ [1]. There is at present a definite interest, particularly in Frascati, in the realization of electron-positron colliding beams. In this note we shall briefly present some further theoretical considerations on high energy electron-positron experiments.

2. – High energy e^+-e^- experiments can test the validity of quantum electrodynamics at small distances. There are two other aspects of such experiments that we want to stress:

i) The possibility of exploring form factors of strong interacting particles. These form factors are explored for timelike momentum transfers. Electron scattering experiments — whenever possible — can only explore spacelike momentum tranfers.

ii) The possibility of carrying out consistently a « Panofsky program », *i.e.* the exploration of the spectrum of masses of elementary particles through their interaction with photons. This program can be extended to include the exploration of particular classes of unstable states.

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[1] N. Cabibbo and R. Gatto: *Phys. Rev. Lett.*, **4**, 313 (1960). The same results have also been derived by Yung Su Tsai: *Phys. Rev.*, **120**, 269 (1960).

3. – Perturbation theory results.

We first list some relevant perturbation theory results. We consider the lowest order graph

Fig. 1.

where P is a charged spin $\frac{1}{2}$ fermion (f) different from the electron, or a charged spin zero boson (b), or a charged spin 1 boson (B). We call θ the c.m. production angle, α the fine structure constant, $\lambda\!\!\!^{-}$ the wavelength and E the energy of each incident particle in the c.m. system, m_i and β_i the mass and velocity of particle i in the c.m. system. The cross sections are

a) for $e^+ + e^- \to f^+ + f^-$:

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi}{4}\alpha^2 \lambda\!\!\!^{-2}\beta_f\left[\frac{1}{2}(1+\cos^2\theta) + \frac{m_f^2}{2E^2}(1-\cos^2\theta)\right],$$

$$\sigma_{\text{total}} \simeq \frac{1}{m_f^2}(2.1\cdot 10^{-32}\ \text{cm}^2) f(x), \qquad (m_f \text{ in GeV}),$$

with

$$f(x) = \frac{1}{x^2}\left(1-\frac{1}{x^2}\right)^{\frac{1}{2}}\left(1+\frac{1}{2x^2}\right),$$

and

$$x = \frac{E}{m_f};$$

b) for $e^+ + e^- \to b^+ + b^-$

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi}{16}\alpha^2 \lambda\!\!\!^{-2}\beta_b^3 \sin^2\theta,$$

$$\sigma_{\text{total}} = \frac{1}{m_b^2}(0.5\cdot 10^{-32}\ \text{cm}^2) b(x), \qquad (m_b \text{ in GeV}),$$

with

$$b(x) = \frac{1}{x^2}\left(1 - \frac{1}{x^2}\right)^{\frac{3}{2}},$$

and

$$x = \frac{E}{m_b};$$

c) for $e^+ + e^- \to B^+ + B^-$

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi}{16}\alpha^2 \lambda\!\!\!^{-2} \beta_B^3 \left[2\left(\frac{E}{m_B}\right)^2 (1 + \cos^2\theta) + 3\sin^2\theta\right],$$

$$\sigma_{total} = \frac{1}{m_B^2}(2.1 \cdot 10^{-32}\ \text{cm}^2) B(x), \qquad (m_B \text{ in GeV}),$$

with

$$B(x) = \frac{3}{4}\left(1 - \frac{1}{x^2}\right)^{\frac{3}{2}}\left(\frac{4}{3} + \frac{1}{x^2}\right),$$

and

$$x = \frac{E}{m_B}.$$

The functions $f(x)$, $b(x)$ and $B(x)$ are reported in Fig. 2. These perturbation theory results are valid only as long as P does not have strong interactions and neglecting radiative corrections.

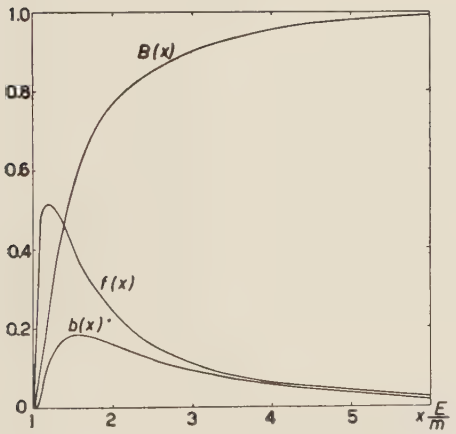

Fig. 2.

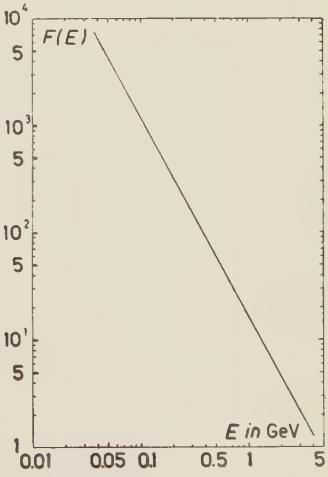

Fig. 3.

The total cross-section for $e^+ + e^- \to \gamma + \gamma$ is given by

$$\sigma_{total} \simeq (3.2 \cdot 10^{-32}\ \text{cm}^2) F(E),$$

where $F(E)$, with E in GeV, is reported in Fig. 3.

4. – Production of strong-interacting particles.

We consider the lowest order graph in the electromagnetic interaction but including all strong interaction effects.

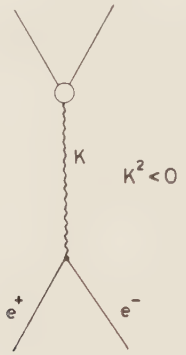

Fig. 4.

The form of the $PP\gamma$ vertex is only limited by Lorentz- and gauge-invariance.

a) for $e^+ + e^- \to f + \bar{f}$ where f is a charged or neutral fermion of spin $\frac{1}{2}$

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi}{8}\alpha^2 \hat{\lambda}^2 \beta_f F(\cos\theta) ,$$

where

$$F(\cos\theta) = |F_1^{(f)}(-4E^2) + \mu_f F_2^{(f)}(-4E^2)|^2(1+\cos^2\theta) +$$
$$+ \sin^2\theta \left| \frac{m_f}{E} F_1^{(f)}(-4E^2)|^2 + \frac{E}{m_f}\mu_f^* |F_2^{(f)}(-4E^2) \right|^2 .$$

Here μ_f is the static anomalous magnetic moment of f, and $F_1^{(f)}(k^2)$, $F_2^{(f)}(k^2)$ are the analytical continuation of the electric and magnetic form factors of f for negative values of k^2. The situation is illustrated in the following graph for the special case of the isotopic vector part of the nucleon electromagnetic vertex.

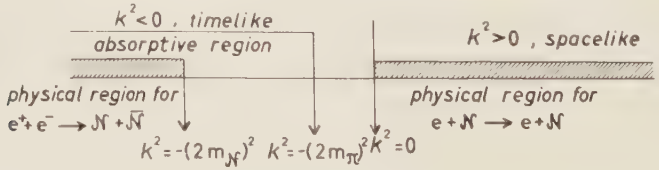

Fig. 5.

In the graph we have reported the physical regions and the absorptive region on the k^2 real axis.

The form factors F_1 and F_2 have an imaginary part along the absorptive cut. Therefore one can have a polarization of the fermion in $e^+ + e^- \to f + \bar{f}$, normal to the production plane (there can be no polarization of the fermion, neglecting radiative corrections, in the scattering $e + f \to e + f$, as follows from time reversal arguments). The polarization $P(\theta)$ is along $\boldsymbol{p}_f \wedge \boldsymbol{p}_{e^+}$ and is given by

$$F(\theta) P(\theta) = -\sin(2\theta) \beta_f^2 \frac{E}{m_f} \operatorname{Im} \{F_1^{*(f)}(-4E^2) \mu_f F_2^{(f)}(-4E^2)\} .$$

The polarization of $\bar{f}$ is $-P(\theta)$ as follows directly from TCP.

b) In $e^+ + e^- \to b + \bar{b}$ where b is a charged or neutral boson of spin zero

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi}{16} \alpha^2 \lambda\!\!\!^{-2} \beta_f^3 |F(-4E^2)|^2 \sin^2\theta ,$$

where $F(k^2)$ is the form factor of b. In general the $\gamma P \bar{P}$ vertex is described in terms of an isotopic vector amplitude and an isotopic scalar amplitude (for pion there is only the vector amplitude, for Λ only the scalar amplitude). The absorptive cut starts at $k^2 = -(2m_\pi)^2$ for the isotopic scalar amplitude, and at $k^2 = -(3m_\pi)^2$ for the isotopic vector amplitude, except for the $\gamma \Sigma \Sigma$ vertex, for which the $\Sigma \to \Lambda + \pi$ transformation produces a lowering of the threshold (2).

c) For $e^+ + e^- \to B + \bar{B}$ where B is a neutral or charged spin one meson

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi}{16} \alpha^2 \lambda\!\!\!^{-2} \beta^3 \left\{ 2 \left(\frac{E}{m_B}\right)^2 |F_1(-4E^2) + \mu F_2(-4E^2) + \varepsilon F_3(-4E^2)|^2 (1 + \cos\theta) + \right.$$

$$\left. + \sin^2\theta \left| 2 \left| F_1(-4E^2) + 2 \left(\frac{E}{m_B}\right)^2 \varepsilon F_3(-4E^2) \right|^2 + \left| F_1(-4E^2) + 2 \left(\frac{E}{m_B}\right)^2 \mu F_2(-4E^2) \right|^2 \right] \right\} ,$$

where F_1, F_2 and F_3 are form factors, μ is the anomalous magnetic moment and ε is the quadrupole moment of B.

5. – Effects of the proposed $T = 1$, $J = 1$ pion-pion resonance.

Except for $e^+ + e^- \to \pi^+ + \pi^-$ and $e^+ + e^- \to \pi^+ + \pi^- + \pi^0$ the threshold for production is at higher k^2 than the threshold of the absorptive region. This circumstance will make theoretical predictions very difficult. The reactions $e^+ + e^- \to \pi^+ + \pi^-$ and $e^+ + e^- \to \pi^+ + \pi^- + \pi^0$ have already been discussed (1). The pion-pion resonance, in the form proposed by Frazer and Fulco (3), would produce a cross-section $\sim 8.3 \cdot 10^{-31}$ cm^2, 17 times bigger than the perturbation theory value at the resonance.

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(2) R. Karplus, C. Summerfield and. E. Wichman; *Phys. Rev.*, **111**, 1187. (1958).
(3) W. A. Franzer and. I. R. Fulco: *Phys. Rev.*, **117**, 1609 (1960).

A favorable process, from the point of view of theoretical analysis, is $e^+ + e^- \to \to \pi^0 + \gamma$, as shown in the following graph where the different regions along the k^2 axis, for the vertex $\pi^0 \to \gamma + \gamma$, when one γ has mass $-k^2$ are indicated.

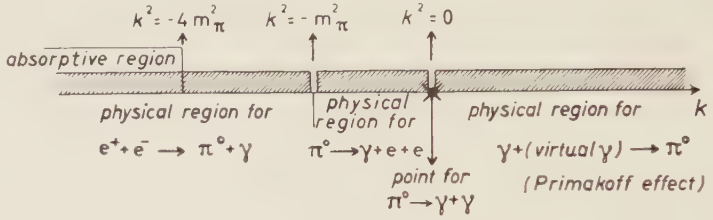
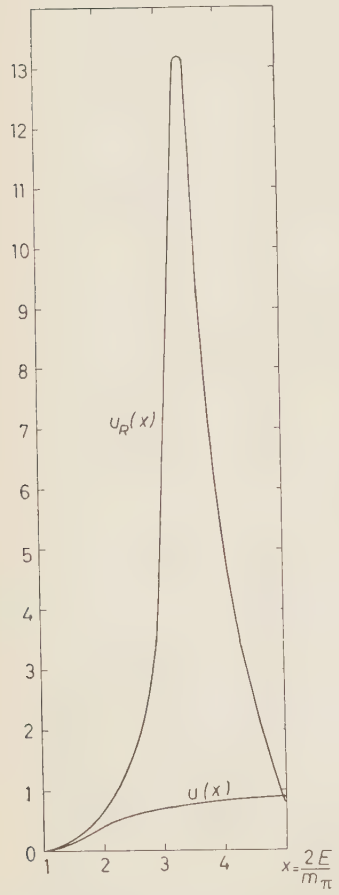


Fig. 6.

The cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)} = \pi\alpha \frac{\tau^{-1}}{m_\pi^3} \beta_\pi^3 (1 + \cos^2\theta) \left| \frac{G(-4E^2)}{G(0)} \right|^2,$$



where $G(k^2)$ is the form factor for $\pi^0 \to \gamma + \gamma$. τ^{-1} is the inverse of the $\pi^0 \to \gamma + \gamma$ lifetime and it is proportional to $|G(0)|^2$. We have used dispersion relations to calculate $G(-4E^2)$, assuming that only the resonant $T=1$, $J=1$, 2π state contributes to the absorptive part and using the $\gamma + \pi \to \to \pi + \pi$ amplitude given by WONG [4]. The result is shown in Fig. 7. The total cross-section for $e^+ + e^- \to \pi^0 + \gamma$ is given by $2.75 \cdot 10^{-35}$ cm^2 $u_R(x)$ with the resonance model. In perturbation theory $(G \equiv G(0))$ it is given by $2.75 \cdot 10^{-35}$ cm$^2 \cdot u(x)$. We have assumed $2.2 \cdot 10^{-16}$ s for the π^0 lifetime. For the charged pion form factor we have used the form proposed by FRAZER and FULCO. With the form proposed by BOWCOCK, COTTINGHAM and LAURIÉ the maximum of $u_R(x)$ would be at higher x ($x \sim 4.8$) and 2.1 times bigger.

Fig. 7. – The cross section for $e^+ + e^- \to \pi^0 + \gamma$ is $2.75 \cdot 10^{-35}$ $u_R(x)$ cm^2 with Frazer-Fulco resonance; it is $2.75 \cdot 10^{-35} u(x)$ if the energy dependence of the form factor is neglected. The assumed π^0 lifetime i.s $2.2 \cdot 10^{-16}$ s.

(4) A. WANG; *Phys. Rev. Lett.*, **5**, 70 (1960).

6. – General discussion of the possible resonances. Detection of Nambu's neutral vector meson.

In the last section we discussed two possibilities of direct detection of the suggested $T=1$, $J=1$, π-π resonance from e^+-e^- collisions. In this section we shall show that e^+-e collisions are very suitable for detecting possible neutral resonant states with $J=1$ and charge conjugation number $C=-1$ (and of course, with zero nucleonic number and zero strangeness). We consider the effect of the resonance in the reaction $e^+ + e^- \to$ (final state) to originate from a graph of the type

Fig. 8.

where the resonant state of mass M and spin J (0 or 1) decays into the final state f with a branching ratio B_f of its total rate Γ. The cross-section curve has to be folded with the experimental resolution curve that we approximate with a rectangle of width $2\Delta E$. We consider three cases: *a*) narrow resonance, $2\Delta E > \Gamma$; *b*) wide resonance $2\Delta E < \Gamma$; *c*) intermediate case, $2\Delta E \approx \Gamma$. The contribution to $e^+ + e^- \to$ $\to$ (final state), from the resonance, for an experiment at the resonance with spread ΔE is then approximately given: for *a*) by $\sigma_{av} \simeq 2\pi\lambda\!\!\!^{-2}(\pi/4)(2J+1)B_i B_f(\Gamma/2\Delta E)$; for *b*) by $\sigma_{max} = \pi\lambda\!\!\!^{-2}(2J+1)B_i B_f$; for *c*) both formulae can safely be applied since they coincide in this case apart from a factor $\pi/2$. A typical comparison can be made for instance with $e^+ + e^- \to \mu^+ + \mu^-$, the cross-section of which is $(\pi/3)\alpha^2\lambda\!\!\!^{-2}$ for $E \gg m_\mu$. Practically, for intensities of the order $(10^{10} \div 10^{11})$ electrons or positrons, the discussion can be limited to those decay modes for which $B_i \simeq 1$. Therefore the relevant quantities are: $\Gamma_i/\Delta E$ in case *a*); Γ_i/Γ in case *b*) and, of course, any of these two quantities in case *c*). From charge conjugation and gauge invariance one sees that Γ_i (rate into $e^+ + e^-$) is proportional to: $\alpha^4 m_e^2$ for $J=0$, $C=1$; $\alpha^6 m_e^2$ for $J=0$, $C=-1$; α^4 for $J=1$, $C=1$; α^2 for $J=1$, $C=-1$.

It seems reasonable to assume $\Delta E \sim 1$ MeV. Then in case *a*) only resonances $J=1$, $C=-1$ would produce relatively large effects. The same conclusion applies also to cases *b*) and *c*) since $\Gamma \geqslant \Delta E$ implies a fast decay by strong interaction or at most by single γ emission ($\Gamma \propto \alpha$). A $J=1$, $C=-1$ meson with $T=1$ can decay rapidly into pions. The resonant effects discussed in the last chapter belong to this type. A $J=1$, $C=-1$ meson with $T=0$, called ρ^0, has been pro-

posed in particular by NAMBU (5), and calculations have been made by HUFF (6). We don't assume any direct coupling of ρ^0 with electrons. Following HUFF we examine in detail three special cases:

1) $m_\rho = m_\pi$: only $\rho^0 \to e^+ + e^-$ is possible, with $\Gamma \simeq 10^{16}\ s^{-1}$, giving for $e^+ + e^- \to \rho^0 \to e^+ + e^-$ a $\sigma_{av} \simeq 7.8 \cdot 10^{-30}\ cm^2$. The cross-section for $e^+ + e^- \to \gamma + \gamma$ at this energy is $\simeq 7.5 \cdot 10^{-29}\ cm^2$.

2) $m_\rho = 2m_\pi$: $\rho^0 \to \pi^0 + \gamma$, $\to e^+ + e^-$, $\to \mu^+ + \mu^-$ are possible with $B_{\pi^0\gamma} \simeq 1$, $B_{e^+e^-} \simeq B_{\mu^+\mu^-} \simeq 10^{-2}$ and $\Gamma \simeq 1.5 \cdot 10^{-19}$ s. Then for $e^+ + e^- \to \rho^0 \to \pi^0 + \gamma$, $\sigma_{av} \simeq 1.3 \cdot 10^{-29}\ cm^2$, and for $e^+ + e^- \to \rho^0 \to e^+ + e^-$, or $\to \mu^+ + \mu^-$, $\sigma_{av} \simeq 10^{-31}\ cm^2$; to be compared with the cross-section for $e^+ + e^- \to \mu^+ + \mu^-$, at that energy $\simeq 0.86 \cdot 10^{-30}$. The perturbation theory estimate for $e^+ + e^- \to \pi^0 + \gamma$ gives $2.3 \cdot 10^{-35}\ cm^2$.

3) $m_\rho = 3m_\pi$: we assume rates $\sim 10^{21}\ s^{-1}$ for $\rho^0 \to \pi^0 + \gamma$, $\to 2\pi + \gamma$ and $B \simeq 10^{-3}$ for $\rho^0 \to e^+ + e^-$, $\to \mu^+ + \mu^-$. The main effect is in $e^+ + e^- \to \pi^0 + \gamma$ or $2\pi + \gamma$ giving $\sigma_{av} \simeq 10^{-29}\ cm^2$, to be compared with the perturbation theory estimate $\simeq 3.8 \cdot 10^{-35}\ cm^2$ for $e^+ + e^- \to \pi^0 + \gamma$ at this energy.

7. – Problems concerning weak interactions. Detection of the intermediate vector meson.

We pointed out the relevance of $e^+ + e^-$ experiments for a Panofsky program. A charged fermion f weakly interacting and with mass $> m_K$ would hardly have been detected if there is a selection rule forbidding the mixed couplings (feγ) or (f$\mu\gamma$). A very interesting possibility is the detection of semiweak-interacting boson B with mass $> m_K$, as suggested, for instance, for mediating weak interactions. The relevant cross-sections are given in Section 3. If B has an anomalous magnetic moment μ_B the differential cross section becomes

$$\frac{d\sigma}{d(\cos\theta)} = \frac{\pi}{16}\alpha^2\lambda^2\beta^3\left\{2(1+\mu_B)^2\left(\frac{E}{m_B}\right)^2(1+\cos^2\theta) + \left(2+\left(\frac{E}{m_B}\right)^1(1-\beta^2+2\mu_B)^2\right)\sin^2\theta\right\}.$$

Note that this formula gives a cross-section that goes to a constant for $E \gg m_B$ if $\mu_B = 0$, or increases $\propto E^2$ if $\mu_B \neq 0$. In both cases unitarity is violated at high energy. From angular momentum and charge conjugation considerations the cross-section should not exceed $2\pi\lambda^2$ if unitarity is respected, neglecting radiative corrections. In fact in this case only the $^3S_1$ and $^3D_1$ waves can interact, giving a maximum interaction of $2\pi\lambda^2$ after summing over polarizations. It follows that, even for $\mu_B = 0$, unitarity is violated at high energy (precisely for $E > 340 m_B$). Another interesting possibility is to look for a semiweakly coupled B^0, which would give rise to resonances in $e^+ + e^- \to e^+ + e^-$, $\to \mu^+ + \mu^-$, $\to \pi^+ + \pi^-$, etc. The formulae of the preceding section, case *a*) can then be applied. On the basis of the semiweak coupling we assume $\Gamma \simeq 5 \cdot 10^{17}\ s^{-1}$ with branching ratios $\sim \frac{1}{5}$ for $B^0 \to e^+ + e^-$, or $B^0 \to \mu^+ + \mu^+$. For $e^+ + e^- \to X^0 \to \mu^+ + \mu^-$ one has then $\bar{\sigma}_{av} \simeq 2\pi\lambda^2(2.6 \cdot 10^{-5})$ which is about 3 times bigger than $e^+ + e^- \to \mu^+ + \mu^-$ with $\sigma \simeq 2\pi\lambda^2(0.88 \cdot 10^{-5})$ for $E \gg m_\mu$.

(5) Y. NAMBU; *Phys. Rev.*, **106**, 1366, (1957).
(6) R. W. HUFF: *Phys. Rev.*, **112**, 1021. (1958).

The contribution from the local weak interactions to, for instance, $e^+ + e^- \to \mu^+ + \mu^-$ is very small at low energy but increases fast with energy. The addition to the lowest order electromagnetic matrix element of a term

$$(2\pi)^{-1}\,(\sqrt{8}G)[\bar{u}(\mu^-)\gamma_\mu \tfrac{1}{2}(1+\gamma_5)v(\mu^+)][\bar{v}(e^+)\gamma_\mu \tfrac{1}{2}(1+\gamma_5)u(e^-)]\,,$$

modifies the cross-section formula to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}(\cos\theta)} = \frac{\pi}{8}\,\alpha^2\lambda\,[(1+\cos^2\theta)(1+\varepsilon+\varepsilon^2) + 2(\varepsilon+\varepsilon^2)\cos\theta]\,,$$

where $\varepsilon \simeq 6.2\cdot 10^{-4}(E/m_N)^2$, with m_N = nucleon mass, and we have taken for G the value of the Fermi constant. A $\cos\theta$ term in the differential cross-section also arises from graphs with two photons exchanged. More uniquely a longitudinal polarization of the $\mu^\pm$ given by

$$P^\pm = \pm\,(\varepsilon+\varepsilon^2)\,\frac{(1+\cos\theta)^2,}{(1+\cos^2\theta)+(\varepsilon+\varepsilon^2)(1+\cos\theta)^2}\,,$$

would indicate the presence of weak interactions. However ε becomes ~ 1 only for colliding beams each of ~ 30 GeV.

* * *

It is our pleasure to acknowledge the stimulating discussions on the subject at Frascati with C. Bernardini, G. Ghigo, and in particular with B. Touschek.

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A Note on the Deuteron Photodisintegration.

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(ricevuto l'8 Febbraio 1961)

The extrapolation procedure from physical region to an unphysical point has been recently extensively used in investigating elementary particle interactions; the results have been fairly good, leading for instance to the determination of the pion-nucleon coupling constant. The question arises as to whether such a procedure could be applied to the physics of the composite particles, in particular to the field of nuclear physics.

The problem of the deuteron photodisintegration is characteristic in this respect; in fact the deuteron can be considered as an elementary particle acting as a source of nucleons; a coupling constant N can then be defined and its value deduced by an extrapolation procedure. The deuteron has been chosen because it has a well known structure, so that the extrapolation can be a test of the analitic properties of the scattering amplitude. Indeed the value of N obtained by this procedure can be compared with the accepted value derived from the theory in first quantization.

Our approximation is very rough since we consider non relativistic nucleon currents only. This first step was rather a preliminary check on the validity of the extrapolation procedure. The present state of experiments and the uncertainties in the extrapolation do not yield much hope of including the D-wave contribution and other corrections in any significant way. In spite of this, the result has been surprisingly good, though affected by a large error.

We use the kinematical variables as defined in [1], formulae (36) and (37). Let $x=\cos\theta$ and let us study the singularities of the transition amplitude in the x-plane, as they appear in the perturbation expansions [2]. This amplitude has two poles in $x=E/q=x_P$ and $x=-x_P$, and two cuts with anomalous thresholds in $x=\pm x_0$, where

$$x_0 = (E/q) + \mu(\mu + 2\alpha)/kq \,. \tag{1}$$

[1] V. De Alfaro and C. Rossetti: *Nuovo Cimento*, **18**, 783 (1960).
[2] See the Mandelstam representation proposed in [1].

The T-matrix for the photodisintegration process can then be written as

$$T = T_P + (\text{R.T.}) , \tag{2}$$

where T_P is the contribution of the single pole in $x = x_P$, and (R.T.) stands for the remainder which is regular at that point. Then the function

$$F(x, \omega) = (1 - x/x_P)^2 \, \mathrm{d}\sigma/\mathrm{d}\Omega \quad (^3) , \tag{3}$$

is regular in $x = x_P$, and assumes there the value $R(\omega)$.

Our task is now to compute theoretically $R(\omega)$, in order to compare its value whith those obtained by extrapolation from experimental angular distributions. Let $H_{\gamma p}$ be the photon-proton interaction hamiltonian. Then, dropping the spin indexes, one can write

$$T_P = \langle q_1 q_2 | H_{\gamma p} | p_1 \rangle = \langle q_2 | H_{\gamma p} | p_1 - q_2 \rangle \Phi\{(\boldsymbol{q}_1 - \boldsymbol{p}_1)/2\} , \tag{4}$$

where $\Phi(\boldsymbol{n})$ is the Fourier transform of the deuteron wave function. This is the contribution of the graph in which the proton only interacts with the γ-ray, the neutron acting as a spectator. If the asymptotic behaviour of the deuteron wave-function is $\varphi_{\mathrm{d}}(r) \underset{r \to \infty}{\simeq} N \exp[-\alpha r/r]$, then $\Phi(\boldsymbol{n})$ has the form

$$\Phi(\boldsymbol{n}) = \frac{4\pi N}{\alpha^2 + n^2} + (\text{R.T.}) , \tag{5}$$

where (R.T.) means terms regular in $n^2 = -\alpha^2$. In the c.m. variables the singular part of Φ can be written as

$$\Phi_P\{(\boldsymbol{q}_1 - \boldsymbol{p}_1)/2\} = \frac{4\pi N}{kE(1 - xq/E)} . \tag{6}$$

The matrix element of $H_{\gamma p}$ is easily calculated in the non-relativistic approximation (2-dimensional spinors)

$$\langle q_2 | H_{\gamma p} | p_1 - q_2 \rangle = -(4\pi \cdot k)^{-\frac{1}{2}} (e/m) \{(\boldsymbol{q} \cdot \boldsymbol{\epsilon}) + i \boldsymbol{\sigma}_p \wedge \boldsymbol{k} \cdot \boldsymbol{\epsilon} \mu_p/2\} , \tag{7}$$

where $\boldsymbol{\epsilon}$ is the photon polarization vector.

The cross-section is then

$$\mathrm{d}\sigma/\mathrm{d}\Omega = (e^2/8\pi m) \left\{ q^2(1 - x) + \mu_p^2 \frac{k^2}{2} \right\} \frac{q}{k} \Phi_p^2 + (\text{R.T.}) \Phi_p + (\text{R.T.}) . \tag{8}$$

It follows that

$$R(\omega) = \left\{ \left(1 - \frac{q}{E} x\right)^2 \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right\}_{x = E/q} = -C \left(1 - \frac{\mu_p^2}{2} \frac{k^2}{m^2}\right) \frac{q m^2}{k^3} \frac{1}{E} , \tag{9}$$

(3) ω is the photon energy in the laboratory system.

where $C=2\pi N^2 e^2/m$. The theoretical value of C is known at once from

$$N^2 = \frac{1}{4\pi} \frac{2\alpha}{1-\alpha r_0},$$

where r_0 is the triplet effective range (strictly speaking, if we take into account the D-state contribution N^2 is no longer a constant). The value obtained for C is $0.587 \cdot 10^{-3}$.

Our proposal is now deduce a value of C by averaging on the extrapolation of experimental data. From formula (1) it is seen that at the threshold

$$x_P \sim mq^{-1}; \qquad x_0 - x_P \sim (m\mu(\mu+2\alpha)/\alpha^2)q^{-1},$$

and for large q,

$$x_P \sim 1 + (m^2/2)q^{-2}; \qquad x_0 - x_P \sim \mu(\mu+2\alpha)q^{-2}.$$

We may then have some success in extrapolating at intermediate energies; at low energy we have to reach a point very far from the physical region, whereas at high energy also the cut has a strong influence on the physical region.

We have considered, as experimental data, the results by WHALIN, SCHRIEVER and HANSON (4), and KECK and TOLLESTRUP (5). A best fit of the experimental points, multiplied by $(1-xq/E)^2$ has been done, and then extrapolation in $x=E/q$. The experimental angular distributions are however affected by rather large errors, and the number of points for fixed energy is only between 4 and 6. We have thus been compelled to search for a best fit with second order polinomials, although the result will certainly be inadequate, because higher order curves were inpractical to use (6). In spite of this shortcomings the results were not bad. The values obtained in this way for $R(\omega)$ (we call them $R_{ex}(\omega)$) are shown in Table I. Here the starred values are those corresponding to angular distributions which at glance could not be fitted by a second order curve, showing clearly a maximum and minimum.

A best fit of the obtained values of $R_{ex}(\omega)$ has been done by a formula of type (9); *i.e.* by putting

$$R_{ex}(\omega) = -C_{ex}\varphi(\omega),$$

where

$$\varphi(\omega) = \left\{1 - \frac{\mu_D^2}{2}\frac{k^2}{m^2}\right\}\frac{qm^2}{k^3}\frac{1}{E^2}.$$

With this procedure the results obtained for various sets of data are shown in

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(4) E. A. WHALIN, B. D. SCHRIEVER and A. O. HANSON: *Phys. Rev.*, **101**, 377 (1956), later on referred to as Ill.

(5) J. C. KECK and A. V. TOLLESTRUP: *Phys. Rev.*, **101**, 360 (1956), later on referred to as Cal.

(6) Due to the scarce number of the experimental points, we could not attempt an analogous extrapolation for $x=-x_P$, where the residue is smaller and of the opposite sign with respect to that in x_P.

Table II. The value of C given by all the data is about $0.41 \cdot 10^{-3}$; if however we exclude the starred data, we obtain as the most probable value $C_{ex} = 0.57 \cdot 10^{-3}$.

Table I.

ω(MeV)	$x_p = E/q$	$-R_{ex}(\omega)/\mu b$	Number of experimental points	Reference	$-R(\omega)/\mu b$
65	4.01	202	5	Ill.	204
80 *	3.62	7.6	5	Ill.	122
105 *	3.19	10.3	5	Ill.	62.3
114 *	3.08	8.4	4+1	Ill.	49.5
140	2.78	22.1	5	Ill.	29.2
149	2.72	15.2	5	Ill.	25.5
194	2.43	26	5	Ill.	12.8
248 *	2.19	3.12	5	Ill.	6.42
105	3.19	41	4	Cal.	62.3
155	2.67	14.4	6	Cal.	23.1
205	2.36	9.4	6	Cal.	10.6
255	2.18	9.2	6	Cal.	5.88
305	2.03	13.0	6	Cal.	3.40
355	1.91	8.2	6	Cal.	2.08
405	1.82	3.47	6	Cal.	1.25
455	1.75	2.25	6	Cal.	0.743

Obviously the error affecting this result is large; the agreement with the theoretical value is however surprisingly good, and should not be dismissed as insignificant.

Table II.

$C_{ex} \cdot 10^3$	Data
0.41	Ill.
0.58	Ill. — *
0.40	Cal.
0.41	Cal. + Ill.
0.56	Cal. + Ill. — *

We plan to improve our calculations and to extend them to other cases as soon as new measurements of the angular distribution will be known.

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Collective Motions in a Plasma due to Magnetic Interaction.

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It is well known that many-particle systems, interacting *via* Coulomb forces, exhibit a peculiar form of collective motions, the plasma oscillations. This property of the system essentially is due to the net mutual repulsion between particles coming about, if the local density exceeds the average one. It is expected, however, that the inclusion of the magnetic interaction between the moving particles reveals other possible collective modes. Indeed, the fact that parallel moving current-elements attract each other and thus local current fluctuations tend to increase, indicates an instability rather than oscillation.

The existence of such collective modes, due to the static magnetic interaction between the current elements represented by the moving electrons, has been investigated within the random phase approximation [1], and a preliminary report is given in this letter. The effect of the electric forces (Coulomb and induction field) has not been included. This neglect is justified on the basis of the following considerations: 1) the neglect of the fluctuating electric field amounts to the neglect of collisions, which are rare events in comparison with the time scale of the collective motions; 2) the average field which is caused by the plasma oscillations can be omitted because, in a good approximation, plasma oscillations being longitudinal can be decoupled from the magnetic modes which are transverse; 3) the induction field ($\mathrm{rot}\, \boldsymbol{E} = -\dot{\boldsymbol{B}}$) influences only the dispersion relations at small wave numbers but does not alter appreciably the physical picture. (The reason for this can be understood by expressing $\boldsymbol{E}_k$ from the Maxwell equations with the neglect of the displacement current: $\boldsymbol{E}_k = -(\mu_0 e/k^2)\boldsymbol{j}_k$. Substituting this into the equation of motion for v_i, this yields a correction indeed for appreciably small wave-numbers only. This feature is carried through to the corrected dispersion relation.)

The equation of motion for the Fourier component of the density of the particle

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(1) D. Bohm and D. Pines: *Phys. Rev.*, **92**, 609 (1953); D. Bohm: *General theory of collective coordinates. The Many Body Problem*, (Paris, 1958).

current in random phase approximation is

$$(1)\qquad \begin{cases} j^{\mu}_{\boldsymbol{k}} + (\gamma^2)^{\mu\nu} j^{\nu}_{\boldsymbol{k}} = -\sum_i (\boldsymbol{k}\cdot\boldsymbol{v}_i)^2 v_i^{\mu} \exp[-i\boldsymbol{k}\cdot\boldsymbol{x}_i] \\ (\gamma^2)^{\mu\nu} = \varkappa^2[\langle v^{\mu}v^{\nu}\rangle + (k^{\alpha}k^{\mu}/k^2)\langle v^{\alpha}v^{\nu}\rangle - (k^{\alpha}k^{\beta}/k^2)\langle v^{\alpha}v^{\beta}\rangle\,\delta^{\mu\nu}] , \\ \varkappa^2 = \dfrac{e^2 n \mu_0}{m} . \end{cases}$$

(Upper indices denote vector components and the summation convention is used; $\boldsymbol{v}_i$ and $\boldsymbol{x}_i$ are velocities and positions of the i-th particle.) It is easy to show that γ^2 actually comprises two collective modes. One depends on the velocity component parallel to $\boldsymbol{j}_{\boldsymbol{k}}$; this quasi-longitudinal mode (the motion is always transverse in the usual sense, since in our approximation $\boldsymbol{k}\cdot\boldsymbol{j}_{\boldsymbol{k}}=0$) corresponds to the physical picture given above: the diagonal elements of this part of γ are real. The other, quasi-transverse mode, depends on the velocity components perpendicular to $\boldsymbol{j}_{\boldsymbol{k}}$: the corresponding part of γ is purely imaginary. The physical picture here is not so immediate: the oscillation is due to the overshooting of the deflected current elements, if there is a large velocity component perpendicular to the deflection. It is remarkable that in the case of thermal equilibrium and no external magnetic field, the two modes just compensate each other and $\gamma^2=0$. If the velocity distribution is not Maxwellian or each component has its own Maxwell distribution, with different temperatures, this is not the case and in a prefixed direction one of the modes persists. Moreover, a properly oriented magnetic field suppresses any of the modes even in thermal equilibrium. The collective modes may be decoupled from the random individual motion in the random phase approximation following the Bohm-Pines method [1]. Introducing the collective co-ordinate

$$(2)\qquad \xi^{\mu}_{\boldsymbol{k}} = \sum_i \left(\frac{\gamma^2}{\omega^2 - (\boldsymbol{k}\cdot\boldsymbol{v}_i)^2}\right)^{\mu\nu} v_i^{\nu} \exp[-i\boldsymbol{k}\cdot\boldsymbol{x}_i] ,$$

one obtains

$$(3)\qquad \xi^{\mu}_{\boldsymbol{k}} + (\omega^2)^{\mu\nu}\xi^{\nu}_{\boldsymbol{k}} = 0 ,$$

where ω^2 is governed by the dispersion relation

$$(4)\qquad -\varkappa^2 \sum_i \left\{[(\beta_i-\omega)^{-2}]^{\mu\alpha} v_i^{\alpha} v_i^{\nu} - [(\beta_i-\omega)^{-1}]^{\mu\alpha} k^{\alpha} v_i^{\nu} - \beta_i \delta^{\mu\nu})/k^2\right\} = \delta^{\mu\nu} ,$$
$$\beta_i = \boldsymbol{k}\cdot\boldsymbol{v}_i .$$

For a singly-lumped velocity distribution the roots of the diagonal elements of ω are situated either along the imaginary axis or in the neighbourhood of the real axis on the lower half-plane, the absolute values being not very far from the corresponding values of γ — provided that k is smaller than a certain maximum value, in the order of magnitude of $\varkappa$. Otherwise the imaginary roots disappear, and the complex roots move very far from the real axis (in a way similar to the behaviour of roots in the case of Landau for plasma oscillations [2]).

[2] L. D. Landau: *Žurn. Éksp. Teor. Fiz.* **10**, 25 (1946); J. D. Jackson: *Internal Report* GM-TR-0165-00535.

Non-linear effects, which may be described as scattering of the collective modes, play an essential role in the case of imaginary roots. This is the mechanism, how the equilibrium distribution of such « bunches » is established. A detailed communication will be given later.

* * *

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Propagators of a Self-Coupled Spinor Field in Edwards and Lieb's Approximation.

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1. – As is well known, the physical many-body propagators play a central role in quantum field theory. It is also known that in many cases the application of the perturbation theory for calculating the propagators leads to meaningless results. Therefore, it seems to be very important to discuss other, non-perturbative, methods, too.

In the present note the physical one- and two-body propagators of the scalar Fermi self-coupling (which cannot be renormalized in Dyson's sense) are investigated by using their continuous integral representations. As there is a quartic part in the Lagrangian, Edwards-Lieb's non-perturbative method (as the only, [1]) has to be applied for approximating the propagators. The calculations are made in first approximation which is probably the best one.

According to Lieb's procedure we start from the continuous integral representation of the vacuum-functional $\langle 0 \,|\, S \,|\, 0 \rangle$

$$(1) \qquad \langle 0 \,|\, S \,|\, 0 \rangle = \frac{\int \delta\psi \, \delta\bar{\psi} \exp\left[i \left(- \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{d}y \, \bar{\psi}(x) S_F^{-1}(x, y) \psi(y) + \int_{-\infty}^{\infty} \mathrm{d}x (g \bar{\psi}(\bar{\psi}\psi)\psi + \bar{\eta}\psi + \bar{\psi}\eta) \right) \right]}{\int \delta\psi \, \delta\bar{\psi} \exp\left[i \left(- \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{d}y \, \bar{\psi}(x) S_F^{-1}(x, y) \psi(y) + g \int_{-\infty}^{\infty} \mathrm{d}x \, \bar{\psi}(\bar{\psi}\psi)\psi \right) \right]},$$

$$S_F^{-1}(x, y) = -\delta(x - y)(i\gamma^\mu \partial_\mu - m) \,,$$

obtained in the usual manner [2], where η, $\bar{\eta}$ are external spinor sources. The continuous integrals in (1) are defined in Matthews and Salam's sense [3]. It must be

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([1]) E. H. Lieb: *Proc. Roy. Soc.*, A **241**, 339 (1957).
([2]) H. Umezawa: *Quantum Field Theory* (Amsterdam, 1956), chap. 18.
([3]) P. T. Matthews and A. Salam: *Nuovo Cimento*, **2**, 130 (1955).

noted that their definition is not quite correct, it is important, however, that also in the cases which could be investigated by the correct definition (bilinear exponentials) different results have not been obtained [4].

Now, we define the functional V for which

$$\langle 0 \,|\, S \,|\, 0 \rangle = \exp\left[i \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{d}y \, \bar{\eta}(x) V(x, y) \eta(y) \right], \tag{2}$$

and assume that the relations concerning the free many-body propagators are valid for both S_F and V. As it can be seen, V determined by Lieb's method satisfies this condition — at least in first approximation as treated here. Now, applying Edwards and Lieb's idea for (1), we write the expressions containing S_F^{-1} in the form

$$-i \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{d}y \, \bar{\psi} S_F^{-1} \psi \equiv -i \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{d}y \, \bar{\psi} V^{-1} \psi + i \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{d}y \, \bar{\psi} (V^{-1} - S_F^{-1}) \psi , \tag{3}$$

and we expand the function

$$\exp\left[i \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{d}y \, \bar{\psi} (V^{-1} - S_F^{-1}) \psi + ig \int_{-\infty}^{\infty} \mathrm{d}x \, \bar{\psi} (\bar{\psi} \psi) \psi \right].$$

In this manner from (1), (2), (3) follows the definition equation of V. In first approximation (from the first two terms of the series mentioned above) we have for V

$$\left[i\gamma^{\mu} \frac{\partial}{\partial x_1^{\mu}} - m + 2ig(\mathrm{Sp}\, V(1, 1) - V(1, 1)) + g \int_{-\infty}^{\infty} \mathrm{d}x_3 \mathrm{d}x_4 \, \bar{\eta}(3) V(3, 1) V(1, 4) \eta(4) \right] V(1, 2) = \tag{4}$$

$$= -\delta(1 - 2) .$$

2. – For the one-body propagator (4) gives the following

$$\left\{ i\gamma^{\mu} \partial_{\mu}^{2} - [m + 2ig(S'_F(0) - \mathrm{Sp}\, S'_F(0))] \right\} S'_F(x - y) = -\delta(x - y) . \tag{5}$$

Supposing that the bare mass does not vanish, $iS'_F(0)$ is a non-vanishing real number, therefore, (5) can be made finite by mass renormalization. The observable mass is defined by the quantity

$$m_R = m + \delta m = m + 2ig(S'_F(0) - \mathrm{Sp}\, S'_F(0)) . \tag{6}$$

Thus, in case of $m \neq 0$ S'_F is identical with the propagator of a free particle of

[4] W. K. Burton and A. H. de Borde: *Nuovo Cimento*, **4**, 254 (1956).

mass m_R, in which the self-mass δm diverges quadratically

$$\delta m = \frac{3gm_R}{8\pi^2} \lim_{\Lambda\to\infty} \left[\Lambda^2 - m_R^2 \ln\left(\frac{\Lambda^2}{m_R^2} + 1\right)\right]. \tag{7}$$

Comparing these results with the first order perturbation theory, two differences can be seen, we have disregarded that the coupling is weak and the self-mass depends on the observable mass.

There is another situation if the bare mass m is vanishing. In this case, because of the γ_5-invariance, S'_F is identical with the free neutrino propagator.

3. – Finally, the first approximation of the two-body propagator will be investigated. It can be shown by comparing (1) and (2) that

$$\frac{\delta^2 V_{\varrho\sigma}(1, 4)}{\delta\bar{\eta}_\alpha(2)\,\delta\eta_\beta(3)} \equiv V_{\varrho\alpha\beta\sigma}(1, 2, 3, 4),$$

is responsible for the two-body scattering.

Let us turn to the momentum space and define $V(p_1, p_2, p_3, p_4)$ as

$$V(x_1, x_2, x_3, x_4) = \tag{8}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3 \mathrm{d}p_4 V(p_1, p_2, p_3, p_4) \exp\left[-i(p_1x_1 + p_2x_2 - p_3x_3 - p_4x_4)\right],$$

then (4) leads to the integral equation

$$V_{\varrho\alpha\beta\sigma}(p_1, p_2, p_3, p_4) = g(2\pi)^4\,\delta(p_1 + p_2 - p_3 - p_4)(S'_F(p_1)S'_F(p_4))_{\varrho\sigma}(S'_F(p_2)S'_F(p_3))_{\alpha\beta} + \tag{9}$$

$$+ 2ig(2\pi)^4 \int_{-\infty}^{\infty} \mathrm{d}q\, S'_{F\varrho\tau}(p_1)(\delta_{\tau\omega} V_{\nu\alpha\beta\nu}(q, p_2, p_3, q + p_4 - p_1) -$$

$$- V_{\tau\alpha\beta\omega}(q, p_2, p_3, q + p_4 - p_1))S'_{F\omega\sigma}(p_4).$$

The integral eq. (9) can be solved exactly. As the part containing a single γ^μ will vanish from the integral

$$\int_{-\infty}^{\infty} \mathrm{d}q\, S'_F(q)\, S'_F(q - p),$$

we have the solution

$$V_{\varrho\alpha\beta\sigma}(p_1, p_2, p_3, p_4) = g(2\pi)^4\,\delta(p_1 + p_2 - p_3 - p_4) \cdot \tag{10}$$

$$\cdot \frac{(S'_F(p_2)S'_F(p_3))_{\alpha\beta}(S'_F(p_1)S'_F(p_4))_{\varrho\sigma}}{1 - \frac{3}{2}(2\pi)^4 ig \int_{-\infty}^{\infty} \mathrm{d}q\ \mathrm{Sp}\,(S'_F(q)S'_F(q + p_4 - p_1))}$$

According to (10) the scattering characterized by the interaction $g\bar{\psi}(\bar{\psi}\psi)\psi$ in Lieb's first approximation is not a point scattering. The vertex part depending on the relative momentum has been investigated by ABRIKOSOV *et al.* [5], it is a very important circumstance, however, that the asymptotic behaviour of their vertex function is not correct [6]. Hence, it is not at all satisfactory to sum up only the most divergent chain-diagrams, thus, Edwards-Lieb's approximation of the two-body propagator is rather rough [7]. It seems that this is the reason why ghost states have appeared in Lieb's two-body propagator.

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[5] A. A. ABRIKOSOV, A. D. GALANIN, L. P. GORKOV, L. D. LANDAU, I. YA. POMERANCHUK and K. A. TER-MARTIROSYAN: *Phys. Rev.*, **111**, 321 (1958).

[6] T. YOSHIMURA: *Progr. Theor. Phys.*, **23**, 569 (1960).

[7] See also FULTON's remark about PEIERLS' report: *Proc. of the Sixth Annual Rochester Conference* (1956).

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The Energy Parameter in the Dispersion Relation.

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(ricevuto il 23 Febbraio 1961)

1. – Introduction.

For the proof of a single variable dispersion relation a suitable choice of the integration parameter has to be made in order to exploit the causal commutator to the maximum. In case there is no change in mass of the target particle before and after the collision, the appropriate parameter is easily obtained by working in the Breit frame of reference O_B [1]. If the incoming and outgoing target particle has a mass difference, the dispersion relation in terms of the usual kinematic parameter can not be established.

An appropriate parameter is obtained for the single variable dispersion relation in the most general case when all the four particle masses involved in a two particle collision may be different. The important step in the method consists in a proper choice of the frame of reference O_A, because once this is done the correct choice of the parameter becomes obvious.

2. – Frame of reference O_B.

To avoid irrelevant details we study the scattering of spinless particles. The incoming particles have masses and momenta m, p and μ, k and the outgoing particles have masses and momenta m', p' and μ', k'. We shall assume that either $p - p'$ or $p - k'$ is spacelike. To be definite let us take $p - p'$ as spacelike. The scattering amplitude M can be written in the form [2]

$$M = i\int \mathrm{d}^4x \exp\left[\tfrac{1}{2}(k + k')x\right]\langle p' | [J'(\tfrac{1}{2}x), J(-\tfrac{1}{2}x)] | p\rangle \theta(x) \,. \tag{1}$$

There are three independent energy-momentum vectors and hence six independent scalar kinematic parameters. In the Breit frame of reference O_B the components

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[1] R. Oehme and J. G. Taylor: *Phys. Rev.*, **113**, 371 (1959).

[2] H. Lehmann: *Suppl. Nuovo Cimento*, **14**, 513 (1959).

of p and p' are determined by three of these, *viz.* m, m' and Q.

$$p = \{\sqrt{m^2+Q^2}, 0, 0, Q\}, \tag{2}$$

$$p' = \{\sqrt{m'^2+Q^2}, 0, 0, -Q\}. \tag{3}$$

This co-ordinate system is so oriented that the third independent vector $K = \frac{1}{2}(k+k')$ has vanishing 2-component. The other components are

$$\begin{cases} K_0 = \{(p+p')^2\}^{-\frac{1}{2}} K\cdot(p+p'), \\ K_3 = \dfrac{1}{2Q}\{K\cdot(p'-p) - K_0(\sqrt{m'^2+Q^2} - \sqrt{m^2+Q^2})\}, \\ K_1 = (K_0^2 - K_3^2 - K^2)^{\frac{1}{2}}. \end{cases} \tag{4}$$

One might choose K_0, K_3 and K^2 as expressed above as the rest of the three independent kinematic parameters. The dispersion relation in K_0 will follow in the usual manner. However, note that

$$\begin{cases} K\cdot(p'-p) = 2QK_3 + K_0(\sqrt{m'^2+Q^2} - \sqrt{m^2+Q^2}), \\ \qquad i.e. \\ \frac{1}{2}(k^2-k'^2) = 2QK_3 + (\sqrt{m'^2+Q^2} - \sqrt{m^2+Q^2})K_0. \end{cases} \tag{5}$$

If K_0 and K_3 are regarded independent, it would imply that the projectile mass varies with the integration variable K_0. This is absurd and hence the dispersion relation so obtained is devoid of any physical content. On the other hand if K_3 is regarded dependent on K_0 then the dispersion relation cannot be obtained.

3. – Frame of reference O_A.

The desired frame of reference O_A is obtained from O_B by a Lorentz transformation associated with a velocity

$$v = \frac{\sqrt{m^2+Q^2} - \sqrt{m'^2+Q^2}}{2Q}, \tag{6}$$

along the 3-axis. Since $(p-p')$ is spacelike $|v|$ will be less than unity. The three invariant parameters m, m' and Q determine the components of p and $(p-p')$ in O_A also.

$$p = \{A, 0, 0, B\}, \tag{7}$$

$$p - p' = \{0, 0, 0, \sqrt{-(p-p')^2}\}, \tag{8}$$

where

$$A = \frac{\sqrt{m^2 + Q^2} - vQ}{\sqrt{1 - v^2}}, \qquad B = \frac{-v\sqrt{m^2 + Q^2} + Q}{\sqrt{1 - v^2}}, \tag{9}$$

and

$$(p - p')^2 = (\sqrt{m^2 + Q^2} - \sqrt{m'^2 + Q^2})^2 - 4Q^2 . \tag{10}$$

The other three independent kinematic parameters ω, Ω and ζ are suitably chosen to determine the components in O_A of the remaining independent vector, which is chosen to be K:

$$K = \{\omega, \sqrt{\omega^2 - \Omega^2 - \zeta}, 0, \Omega\} . \tag{11}$$

The invariant definitions of these parameters are

$$\zeta = K^2 , \tag{12}$$

$$\Omega = -\frac{K \cdot (p - p')}{\sqrt{-(p - p')^2}} , \tag{13}$$

$$\omega = \frac{K \cdot p + B\Omega}{A} = \frac{1}{2A}[(p + k)^2 + 2\Omega B - (B + \Omega)\sqrt{-(p - p')^2} - \tfrac{1}{4}(p - p')^2 - A^2 + B^2 - \zeta] . \tag{14}$$

4. – Suitability of the parameters.

Following points establish the suitability of the six kinematic parameters m, m', Q, ω, Ω and ζ for the single variable dispersion relation:

a) The variables ω and ζ whose complex domains have to be studied do not occur in the matrix element $\langle p' | [J'(\frac{1}{2}x), J(-\frac{1}{2}x)] | p \rangle$ in the eq. (1).

b) The scattered particle masses μ, μ' do not vary with the dispersion relation integration variable ω. In fact the following relations exist:

$$\mu^2 = \zeta + \tfrac{1}{4}(p - p')^2 + \Omega\sqrt{-(p - p')^2} , \tag{15}$$

$$\mu'^2 = \zeta + \tfrac{1}{4}(p - p')^2 - \Omega\sqrt{-(p - p')^2} . \tag{16}$$

c) As $m' \to m$, $O_A \to O_B$ and we get back the conventional parameters.

d) The dispersion relation for $\zeta_1 < -\Omega^2$ is obtained in the usual way. Further, the analytic continuation in ζ to the physical value can still be tackled via the Jost-Lehmann-Dyson integral representation. There is slight algebraic complication but no additional difficulty in principle.

LIBRI RICEVUTI E RECENSIONI

Libri ricevuti.

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Recensioni.

J. L. Synge – *Relativity. The General Theory*. North Holland Publishing Co., Amsterdam, 1960; pp. XV-505; fl. 55.

Questo libro è il terzo della serie dello stesso autore sulla teoria della relatività (i due primi volumi sono: *Relativity, the special theory* e *The relativistic gas*). L'introduzione contenuta nella copertina esordisce con la constatazione che il libro è di tendenze conservatrici. Non possiamo non associarci a questo giudizio anche se poi è giusto notare i molti pregi del libro, non ultima la precisione del linguaggio, il rigore dell'esposizione e l'introduzione sistematica di algoritmi matematici (tra cui la world-function) dove erano stati finora ignorati dai libri di testo. Mediante questi artifizi la teoria acquista maggiore unità formale e molti calcoli risultano drasticamente semplificati. Con tutto questo non riteniamo il libro adatto ad un

principiante che desideri iniziarsi ai misteri della relatività generale, semmai lo riteniamo utile all'esperto desideroso di completare la propria preparazione e chiarire molti dei punti oscuri che sempre rimangono a chi si occupa di questo ramo della fisica.

Abbiamo detto che il libro è di tendenze conservatrici. Avremmo infatti desiderato vedere più estesamente citati i molti lavori usciti in questi ultimi anni ad opera di Wheeler, Misner, Brill e molti altri, sul problema della quantizzazione del campo Einsteniano anche se, e i detti autori lo riconoscono, siamo ancora lontani dall'aver risolto il problema.

Chiude il libro una utile ed estesissima bibliografia. La stampa è nitidissima ed il testo contiene molti grafici e disegni.

T. Regge

L. V. Groshev, V. N. Lutsenko, A. M. Demidov e V. I. Pelekhov – *Atlas of γ-Ray Spectra from Radiative Capture of Thermal Neutrons.* Tradotto dal russo da J. E. Sykes. Pergamon Press, Oxford, 1959; pp. 198, £ 7.

Si tratta di una raccolta di dati, aggiornata fino al 1958, ottenuti dallo studio della cattura radiativa dei neutroni termici. Si riduce pertanto ad una serie di tabelle e di grafici sugli spettri γ di cattura. Oltre l'energia dei raggi γ di cattura, sono riportati altri dati, quali la sezione d'urto di cattura termica degli isotopi stabili, la vita media degli isotopi radioattivi che si formano nel processo di cattura, il momento angolare totale dei nuclei prima e dopo la cattura (stato fondamentale), l'energia di legame del neutrone nel nucleo finale.

Tre pagine introduttive danno alcune informazioni sulle fonti dei dati raccolti, sui metodi sperimentali usati e sull'uso delle tabelle e dei grafici. La quantità di materiale raccolto e l'importanza attuale dell'argomento meritavano più di tre pagine. Questo atlante sarà senza dubbio molto prezioso per gli specialisti di fisica nucleare delle basse energie, in particolare per chi si occupa di fisica dei neutroni. Forse per questo il prezzo è così alto.

E. Clementel

F. H. Clauser – *Symposium on Plasma Dynamics.* Addison-Wesley Inc., Reading, Mass., 1960; pp. 360, $ 12.50.

Il libro è un resoconto del simposio internazionale sulla dinamica del plasma tenutosi a Woods Hole nel Massachusetts nel giugno 1958.

Al simposio erano stati invitati i fisici più illustri interessati alla fisica del plasma nei suoi vari aspetti: Astrofisica, Magnetoidrodinamica cosmica, Meccanica dei fluidi, Meccanica statistica, Scarica nei gas, Aerodinamica.

Al simposio non era stata presentata alcuna relazione scritta ma, ciascun argomento veniva esposto da un esperto e la relazione era seguita da un'animata discussione.

Francis Clauser si incaricò di pubblicare in un volume i risultati del simposio ed in questo lavoro ebbe come collaboratori i suddetti relatori.

Il lettore troverà l'opera di grande interesse perchè essa gli offre una vasta panoramica su tutte le branche in cui la Fisica del Plasma può suddividersi. Ciascun capitolo è stato curato da fisici di alto livello che hanno saputo, nel breve spazio a disposizione, mettere a fuoco i problemi fondamentali, spesso non ancora risolti, che stanno alla base della materia trattata.

La trattazione è quindi tutt'altro che superficiale e destinata al fisico che si

è già interessato di plasma: egli potrà trovare in questo libro un ausilio, direi unico, per orientarsi nella complessa dinamica dei gas ionizzati, sia di laboratorio che dello spazio interstellare.

Alla fine del libro è riportata una bibliografia aggiornata al 1958.

Dal Simposio di Woods Hole, sono passati più di due anni e si può quindi notare che qualche capitolo (i primi quattro e il settimo), relativo alla dinamica dei plasmi prodotti in laboratorio, è un pò arretrato; comunque ancora valido.

B. Brunelli

H. Jones – *The Theory of Brillouin Zones and Electronic States in Crystals*. North Holland Pub. Co., Amsterdam, 1960; pp. XV-505, fl. 55.

Questo libro espone la teoria degli stati elettronici nei cristalli nell'approssimazione di particelle indipendenti. Le proprietà generali degli autovalori e delle autofunzioni monoelettroniche sono introdotte, nel primo capitolo, per mezzo dello studio dell'equazione di Schrödinger con potenziale periodico unidimensionale. I tre capitoli seguenti considerano le proprietà elettroniche che sorgono come conseguenza dell'invarianza della Hamiltoniana rispetto alle operazioni di simmetria del gruppo spaziale; l'effetto della simmetria traslazionale (funzioni di Bloch e zone di Brillouin in tre dimensioni) è studiato nel II capitolo, l'effetto delle simmetrie puntuali nel III e nel IV. Il metodo delle zone estese per trattare sistemi con un grande numero di atomi nella cella unitaria è sviluppato nel V capitolo. Nel VI sono discussi brevemente alcuni dei metodi per trovare le soluzioni approssimate dell'equazione di Schrödinger. L'ultimo capitolo infine tratta gli aspetti più rilevanti della teoria dell'accoppiamento di spin-orbita per gli elettroni nei cristalli.

Come si vede da questo rapido sommario, il libro non pretende di essere un testo generale sulle teorie elettroniche dei solidi; esso tocca, secondo le intenzioni stesse del suo Autore, quegli aspetti della teoria che hanno validità generale, ed insiste piuttosto sugli aspetti matematici del problema che su quelli fisici. La sua lettura presuppone la conoscenza di un testo introduttivo sulla teoria dei solidi, e va completata almeno con lo studio più dettagliato dei vari metodi di risoluzione dell'equazione di Schrödinger e dei limiti di validità dell'approssimazione di elettroni indipendenti; si pensi che non vi compare nemmeno un accenno al concetto di « buca ».

Uno dei pregi del libro sta nel fatto che esso non presuppone nozioni particolari di matematica; i concetti essenziali alla trattazione sono introdotti in modo semplice e chiaro nel testo.

L'esposizione è sempre lucida e rigorosa, ma un po' astratta; le considerazioni qualitative di carattere introduttivo e generale, e così pure gli esempi ed i richiami alla situazione fisica sono ridotti all'essenziale, e questo, senza diminuire l'interesse della opera, rende la lettura piuttosto ardua.

R. Fieschi

L. Dresdner – *Resonance Absorption in Nuclear Reactors*. Pergamon Press, Oxford, 1960; pp. 131, s. 40.

Un'interessante monografia della serie nucleare della Pergamon Press. L'Autore è riuscito a coordinare in poco più di cento pagine una lunga serie di lavori sull'argomento, distribuiti in rapporti e riviste talvolta non facilmente accessibili. In sostanza il problema discusso è il calcolo dell'integrale di risonanza,

il quale viene presentato nelle varie approssimazioni finora formulate sia per i sistemi omogenei che per i sistemi eterogenei. Arricchiscono la monografia diversi grafici e molte tabelle numeriche. Lo stato attuale della teoria è reso chiaramente dall'ultimo capitolo, dove sono confrontate le misure sperimentali degli integrali di risonanza con i valori calcolati.

E. CLEMENTEL

Reactions Between Complex Nuclei, Proceedings of the Second Conference on Reactions Between Complex Nuclei, May 2-4, 1960, Gatlinburg, Tennessee. Edited by A. ZUCKER, F. T. HOWARD and E. C. HALBERT. John Wiley & Sons, Inc., New York, 1960; pp. 319, $ 7.00.

Le ricerche sulla interazione della materia con ioni pesanti accelerati si sono sviluppate in questi ultimi anni ad un punto tale da costituire un capitolo a parte della fisica nucleare. Tali ricerche consentono di affrontare i più importanti problemi della fisica dei nuclei, quali la diffusione, la fissione, le reazioni di transfer, ecc. Le caratteristiche generali di questo metodo di indagine sono essenzialmente due. La prima è connessa con la grande massa del proiettile che consente di ottenere informazioni altrimenti non ottenibili; la seconda riguarda la possibilità offerta dagli ioni pesanti di fondersi con i nuclei del bersaglio per formare una materia nucleare altamente eccitata.

I Rendiconti della « Seconda Conferenza sulle reazioni tra nuclei complessi » contengono, suddivisi in cinque capitoli, numerosi ed interessanti lavori sugli argomenti sopra accennati.

Nel primo capitolo sono trattate le reazioni di transfer, cioè le interazioni di tipo « superficiale » caratterizzate dal trasferimento di nucleoni da un nucleo complesso ad un altro (per esempio la reazione $^{197}Au + ^{14}N \rightarrow ^{198}Au + ^{13}N$). BREIT e collaboratori propongono due interpretazioni del fenomeno. In una il trasferimento del nucleone avviene direttamente mediante semplice « tunneling » dalla buca di potenziale del nucleo urtante a quella del nucleo bersaglio attraverso una regione ad energia cinetica negativa. Nel secondo modello il processo di « tunneling » è preceduto da eccitazione coulombiana e ciò per tener conto del fatto che quest'ultima entra in gioco a distanze maggiori di quelle a cui può aver inizio la penetrazione delle barriere. Seguono poi quattro relazioni a carattere sperimentale riguardanti reazioni di transfer con ioni ^{12}C e ^{14}N.

Il secondo capitolo, che tratta la eccitazione coulombiana, contiene sette relazioni riguardanti esperimenti svolti con ioni di neon, ossigeno, litio ed argon ed una relazione sulla teoria della eccitazione coulombiana multipla (K. ALDER ed A. WINTER).

La parte più interessante di questo capitolo riguarda appunto i processi multipli (due o più eccitazioni successive in un solo urto) in quanto essi sono resi possibili dal particolare tipo di proiettile (nucleo complesso) impiegato. ELBEK, STEPHENS e DIAMOND sono riusciti ad ottenere eccitazioni sestuple e momenti angolari molto elevati (fino a 12).

Il terzo capitolo riguarda la diffusione elastica e contiene nuove comunicazioni di cui alcune a carattere teorico.

Il quarto capitolo tratta diversi argomenti e cioè i transuranici, la fissione e le reazioni con litio a basse energie. Esso contiene una comunicazione sul primo argomento, cinque sulla fissione del bismuto, oro, piombo ed uranio e tre sulle reazioni indotte dal litio. Tutti questi processi sono generati in urti vicini ed alcuni di essi sono caratterizzati dal fatto che il nucleo com-

posto derivante dall'unione di due nuclei complessi (proiettile e bersaglio) può avere un momento angolare eccezionalmente elevato, ($\geqslant 30\ \hbar$). Si hanno in tal modo delle situazioni sperimentali irrealizzabili con gli ordinari proiettili. Particolare interesse offre il fatto che si può studiare la fissione sia direttamente che indirettamente, in quanto con i nuclei complessi si può anche realizzare il processo inverso, almeno approssimativamente. Le esperienze sulle reazioni indotte dal litio a bassa energia hanno messo in chiara evidenza l'utilità del modello a cluster.

L'ultimo capitolo tratta la formazione e le proprietà dei nuclei composti. In esso sono contenute tre comunicazioni a carattere teorico e cinque a carattere sperimentale.

Chiude questi rendiconti una interessante esposizione riepilogativa di E. P. WIGNER.

La lettura di questi rendiconti porta a due importanti considerazioni. La prima, nettamente positiva, riguarda l'impressionante « allargamento di orizzonte » prodotto dall'impiego dei proiettili complessi nello studio della fisica dei nuclei. La seconda, preoccupante, riguarda il fatto che, su un totale di quaranta relazioni, soltanto tre non provengono da laboratori statunitensi. Preoccupante, si diceva, perchè, come qualcuno — forse esagerando — ebbe a dire durante la conferenza, i fisici nucleari che non impiegano proiettili complessi nei loro esperimenti, saranno tra breve considerati i parenti poveri della ricerca.

S. SCIUTI

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